ON RHOADES' OPEN QUESTIONS

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ABSTRACT. In this paper the open questions raised by Rhoades [1] are answered or partially answered.

1. INTRODUCTION

In this paper we adopt the notations and terminologies in Rhoades [1].

Throughout this paper let (X, d) be a complete metric space and T a mapping from X into X . According to the classification in [1], T is called an (m) type mapping if it satisfies the following condition (m) , where $m = 25, 50, 75, 100, 125$:

(25) For any $x, y \in X$, $x \neq y$

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

(50) There exists a positive integer p such that for any $x, y \in X$, $x \neq y$

$$d(T^p x, T^p y) < \max\{d(x, y), d(x, T^p x), d(y, T^p y), d(x, T^p y), d(y, T^p x)\}.$$

(75) There exist positive integers p, q such that for any $x, y \in X$, $x \neq y$

$$d(T^p x, T^q y) < \max\{d(x, y), d(x, T^p x), d(y, T^q y), d(x, T^q y), d(y, T^p x)\}.$$

(100) For any given x there exists a positive integer $p(x)$ such that for every $x \in X$, $x \neq y$

$$d(T^{p(x)} x, T^{p(x)} y) < \max\{d(x, y), d(x, T^{p(x)} x), d(y, T^{p(x)} y), d(x, T^{p(x)} y), d(y, T^{p(x)} x)\}.$$

(125) For any given $x, y \in X$, $x \neq y$, there exists a positive integer $p(x, y)$ such that

$$d(T^{p(x, y)} x, T^{p(x, y)} y) < \max\{d(x, y), d(x, T^{p(x, y)} x), d(y, T^{p(x, y)} y), d(x, T^{p(x, y)} y), d(y, T^{p(x, y)} x)\}.$$

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Concerning the existence of fixed point for these mappings the following open questions were raised by Rhoades [1].

1. If $T \in (25)$ is continuous on X , and $\{T^n(x_0)\}_{n \geq 0}$ has a cluster point for some $x_0 \in X$, does T possess a fixed point?
2. If the answer to (1) is no, then what additional hypotheses on T or X are needed in (25) to guarantee the existence of a fixed point?
3. If T satisfies any of the definitions (95)–(100), does T have a fixed point?
4. If T satisfies any of the definitions (104)–(125), does T have a fixed point?
5. Is the concept of locally contractive extendable to (11), (36), (61) or (111)?

Recently, answers to some of these questions were obtained. Question 1 was answered in the negative in [4]. Question 2 was answered in the first author's paper [2]. The purpose of this paper is to establish sufficient conditions or necessary and sufficient conditions for the definitions (50), (75), (100), and (125) to possess a fixed point. By using these results Rhoades' open questions 3 and 4 are partially answered. At the end of this paper we give an example to show that the answer to 5 is no.

2. ANSWERS TO QUESTIONS 3 AND 4

Definition. Let T be a self-mapping of X . A point $x \in X$ is called a periodic point of T , if there exists a positive integer m such that $T^m x = x$. The least positive integer satisfying this condition is called the periodic index of x .

Theorem 1. Let $T \in (125)$ and let $x \in X$ be a periodic point of T with periodic index k . Then T has a fixed point in $\{T^n x\}_{n \geq 0}$ if and only if for any $T^{n_1} x, T^{n_2} x \in A = \{T^n x\}_{n \geq 0}$, $T^{n_1} x \neq T^{n_2} x$, there exist $T^{n'_1} x, T^{n'_2} x \in A$ such that

$$T^{p(T^{n'_1} x, T^{n'_2} x)}(T^{n'_1} x) = T^{n_1} x, \quad T^{p(T^{n'_1} x, T^{n'_2} x)}(T^{n'_2} x) = T^{n_2} x.$$

If this condition is satisfied, then this x is the unique fixed point of T in X .

Proof. The necessity of condition is obvious.

Sufficiency. Since the periodic index of x is k , then

$$A = \{x, Tx, \dots, T^{k-1}x\}.$$

If $x \neq Tx$, then there exist $T^{n_1} x, T^{n_2} x \in A$, $T^{n_1} x \neq T^{n_2} x$, such that

$$\delta(A) = \max_{0 \leq n, m \leq k-1} \{d(T^n x, T^m x)\} = d(T^{n_1} x, T^{n_2} x).$$

By the assumption of the theorem, there exist $T^{n'_1} x$ and $T^{n'_2} x \in A$ such that

$$T^{p(T^{n'_1} x, T^{n'_2} x)}(T^{n'_1} x) = T^{n_1} x, \quad T^{p(T^{n'_1} x, T^{n'_2} x)}(T^{n'_2} x) = T^{n_2} x.$$

Since $T^{n_1}x \neq T^{n_2}x$, $T^{n'_1}x \neq T^{n'_2}x$. Therefore we have

$$\begin{aligned} \delta(A) &= d(T^{n_1}x, T^{n_2}x) \\ &= d(T^{p(T^{n'_1}x, T^{n'_2}x)}(T^{n'_1}x), T^{p(T^{n'_1}x, T^{n'_2}x)}(T^{n'_2}x)) \\ &< \max\{d(T^{n'_1}x, T^{n'_2}x), d(T^{n'_1}x, T^{n_1}x), d(T^{n'_1}x, T^{n_2}x), \\ &\quad d(T^{n'_1}x, T^{n_2}x), d(T^{n'_2}x, T^{n_1}x)\} \\ &\leq \delta(A), \end{aligned}$$

a contradiction, and we have $x = Tx$.

It is easy to see that x is the unique fixed point of T in X .

Since definitions (25), (50), (75), and (100) are special cases of definition (125), the following results can be obtained from Theorem 1 immediately.

Theorem 2. Let $T \in (50)$. Then T has a fixed point in X if and only if there exists a periodic point $x \in X$ of T . If this condition is satisfied, then this x is the unique fixed point of T in X .

Theorem 3. Let $T \in (100)$ and let $x \in X$ be a periodic point of T . Then T has a unique fixed point in $\{T^n x\}_{n \geq 0}$, if and only if for any $T^{n_1}x \in \{T^n x\}_{n \geq 0}$, there exist $T^{n_0}x \in \{T^n x\}_{n \geq 0}$ such that

$$T^{p(T^{n_0}x)}(T^{n_0}x) = T^{n_1}x,$$

where $p(T^{n_0}x)$ is the positive integer appearing in definition (100).

If this condition is satisfied, then this point x is the unique fixed point of T .

Theorem 4. Let $T \in (75)$ and let $x \in X$ be a periodic point of T . If for any $T^{n_1}x, T^{n_2}x \in \{T^n x\}_{n \geq 0}$, $T^{n_1}x \neq T^{n_2}x$, there exist $T^{n'_1}x, T^{n'_2}x \in \{T^n x\}_{n \geq 0}$, $T^{n'_1}x \neq T^{n'_2}x$ such that

$$T^p(T^{n'_1}x) = T^{n_1}x, \quad T^q(T^{n'_2}x) = T^{n_2}x,$$

where p, q are the positive integers appearing in definition (75), then x is the unique fixed point of T .

We now give some sufficient conditions to guarantee the existence of a fixed point for mapping $T \in (75)$.

Theorem 5. Let $T \in (75)$ and let $x \in X$ be a periodic point of T with periodic index k . Suppose p, q the positive integers appearing in definition (75), also satisfy the following conditions:

(i) $p = p_1k + p_2$, $q = q_1k + q_2$, $0 \leq p_2, q_2 < k$, and p_1 and q_1 are non-negative integers;

(ii) $2|p_2 - q_2| \neq k$.

Then x is the unique fixed point of T in X .

Proof. Suppose x is not the fixed point of T , and let

$$A = \{x, Tx, T^2x, \dots, T^n x, \dots\}.$$

Since the periodic index of x is k ,

$$A = \{x, Tx, \dots, T^{k-1}x\},$$

and the points in A are distinct. Therefore there exist $i, j: 0 \leq i < j < k$, such that

$$\delta(A) = \max_{0 \leq m, n \leq k-1, m \neq n} d(T^m x, T^n x) = d(T^i x, T^j x).$$

Without loss of generality, we can assume that $p_2 \geq q_2$. It is easy to see that for any non-negative integer n , $T^n(A) = A$. Hence there exist $T^{n_1}x, T^{n_2}x \in A$ such that

$$(1) \quad T^i x = T^{p_2}(T^{n_1}x), \quad T^j x = T^{q_2}(T^{n_2}x).$$

Similarly, there exist $T^{n'_1}x, T^{n'_2}x \in A$ such that

$$(2) \quad T^i x = T^{q_2}(T^{n'_1}x), \quad T^j x = T^{p_2}(T^{n'_2}x).$$

Next we prove that at least one of the statements $n_1 \neq n_2, n'_1 \neq n'_2$ is true.

Suppose $n_1 \neq n_2, n'_1 = n'_2$. Since

$$0 \leq i, j, p_2, q_2, n_1, n_2, n'_1, n'_2 < k,$$

it follows from (1) and (2) that there exist $a, a', b, b' \in \{0, 1\}$ such that

$$(3) \quad p_2 + n_1 = ak + i, \quad q_2 + n_2 = bk + j,$$

$$(4) \quad q_2 + n'_1 = a'k + i, \quad p_2 + n'_2 = b'k + j.$$

Since $p_2 \geq q_2, n_1 = n_2, ak + i \geq bk + j$. Since $i < j, a = 1, b = 0$. It follows from (3) that

$$(5) \quad (p_2 - q_2) + (j - i) = k.$$

By virtue of (4) and $n'_1 = n'_2$, it follows that

$$(p_2 - q_2) = (b' - a')k + (j - i).$$

Noting that $0 \leq p_2 - q_2 \leq k - 1, 0 < j - i < k$, it follows from (6) that $b' - a' = 0$, and thus $p_2 - q_2 = j - i$. Substituting into (5), we have $2(p_2 - q_2) = k$, contradicting (ii). For definiteness, assume that $n_1 \neq n_2$. Then $T^{n_1}x \neq T^{n_2}x$, and in view of $T^{p_2}x = T^p x, T^{q_2}x = T^q x$, we have

$$\begin{aligned} \delta(A) &= d(T^i x, T^j x) = d(T^{p_2}(T^{n_1}x), T^{q_2}(T^{n_2}x)) = d(T^p(T^{n_1}x), T^q(T^{n_2}x)) \\ &< \max\{d(T^{n_1}x, T^{n_2}x), d(T^{n_1}x, T^p(T^{n_1}x)), d(T^{n_2}x, T^q(T^{n_2}x)), \\ &\quad d(T^{n_1}x, T^q(T^{n_2}x)), d(T^{n_2}x, T^p(T^{n_1}x))\} \leq \delta(A), \end{aligned}$$

a contradiction. Therefore $x = Tx$. It is easy to see that x is the unique fixed point of T in X .

Remark. Definitions (95)–(99) and (104)–(124) are special cases of definitions (100) and (125), respectively. Hence Theorem 3 and 1, respectively, give a partial answer to questions 3 and 4.

3. ANSWER TO QUESTION 5

Proposition. *The concept of locally contractive is not extendable to definitions (4)–(25), (29)–(50), (79)–(100), (104)–(125).*

Proof. This can be seen from the following counterexample:

Counterexample. Let $X = (-\infty, +\infty)$ and let $\varepsilon > 0$ be any given number. Let $f: X \rightarrow X$ be defined by

$$f(x) = x + \lambda, \quad \lambda > \varepsilon.$$

For any $x, y \in X$, when $0 < |x - y| < \varepsilon$, we have

- (a) $|f(x) - f(y)| = |x - y| < \varepsilon,$
- (b) $|x - f(x)| = \lambda,$
- (c) $|y - f(y)| = \lambda,$
- (d) $|x - f(y)| = \begin{cases} \lambda + |x - y|, & \text{if } y \geq x, \\ \lambda - |x - y|, & \text{if } y < x, \end{cases}$
- (e) $|y - f(x)| = \begin{cases} \lambda - |x - y|, & \text{if } y \geq x, \\ \lambda + |x - y|, & \text{if } y < x. \end{cases}$

It follows from (b), (c) and (d), (e) that

- (f) $|x - f(x)| + |y - f(y)| = 2\lambda,$
- (g) $|x - f(y)| + |y - f(x)| = 2\lambda.$

It follows from (a) and (f) that

$$|f(x) - f(y)| = |x - y| < \varepsilon = \frac{2\lambda\varepsilon}{2\lambda} = \frac{\varepsilon}{2\lambda} \{|x - f(x)| + |y - f(y)|\}.$$

Since $(\varepsilon/2\lambda) \in (0, \frac{1}{2})$, $f \in (4)$, hence it also belongs to (5)–(10) and (21)–(25).

On the other hand, from (a) and (g) we can obtain

$$|f(x) - f(y)| = |x - y| < \varepsilon = \frac{2\lambda\varepsilon}{2\lambda} = \frac{\varepsilon}{2\lambda} \{|x - f(y)| + |y - f(x)|\}.$$

This shows that $f \in (11)$, therefore it also belongs to (12)–(17).

If we take $\lambda > 4\varepsilon/3$, it follows from (a), (f) and (g) that

$$\begin{aligned} |f(x) - f(y)| &= |x - y| < \frac{1}{5}|x - y| + \frac{4\varepsilon}{5} < \frac{1}{5}|x - y| + \frac{3}{5}\lambda \\ &= \frac{1}{5}|x - y| + \frac{3}{20}\{|x - f(x)| + |y - f(y)|\} \\ &\quad + \frac{3}{20}\{|x - f(y)| + |y - f(x)|\}. \end{aligned}$$

Thus $f \in (18)$, and so it also belongs to (19) and (20). Therefore when $\lambda > 4\varepsilon/3$, f satisfies (4)–(25), but has no fixed point in X . Hence the concept

of locally contractive is not extendable to $(25i + m)$, $m = 4, 5, \dots, 25$, $i = 1, 2, 3, 4$.

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