ON THE NUMBER OF SELF-DUAL BASES OF $GF(q^m)$ OVER $GF(q)$

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ABSTRACT. Let $E = GF(q^m)$ be the m -dimensional extension of $F = GF(q)$. We are concerned with the numbers $sd(m, q)$ and $sdn(m, q)$ of self-dual bases and self-dual normal bases of E over F , respectively. We completely determine $sd(m, q)$, en route giving a very simple proof for the Sempel-Seroussi theorem which states that $sd(m, q) = 0$ iff q is odd and m is even. Using results of Lempel and Weinberger and MacWilliams, we can also determine $sdn(m, p)$ for primes p .

1. INTRODUCTION

Let $E = GF(q^m)$ be the m -dimensional extension of $F = GF(q)$, the finite field with q elements. (See Lidl and Niederreiter [15] for background on finite fields.) We recall that the *trace function* $Tr: E \rightarrow F$ is defined by

$$(1) \quad Tr(\alpha) = \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}};$$

it is well known that Tr is a linear mapping from E onto F . Moreover, setting

$$(2) \quad (\alpha, \beta) = Tr(\alpha\beta)$$

defines a nondegenerate symmetric bilinear form on E (over F), called the *trace bilinear form*. The elements $\alpha_1, \dots, \alpha_m \in E$ form a basis of E over F if and only if

$$(3) \quad \det A = \det \begin{pmatrix} \alpha_1 & \cdots & \alpha_m \\ \alpha_1^q & \cdots & \alpha_m^q \\ \vdots & \ddots & \vdots \\ \alpha_1^{q^{m-1}} & \cdots & \alpha_m^{q^{m-1}} \end{pmatrix} \neq 0.$$

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$\bar{\alpha} = \{\alpha_1, \dots, \alpha_m\}$ is called a *trace-orthogonal basis*, iff one has

$$(4) \quad \text{Tr}(\alpha_i \alpha_j) = 0, \quad \text{for } i \neq j, i, j \in \{1, \dots, m\}.$$

If additionally

$$(5) \quad \text{Tr}(\alpha_i^2) = 1 \quad \text{for } i = 1, \dots, m,$$

then $\bar{\alpha}$ is called a *self-dual basis*. Finally, a basis of the form $\alpha_i = \alpha^{q^i}$ ($i = 0, \dots, m-1$) for some $\alpha \in E$ is called a *normal basis*. Both self-dual and normal bases (and, in particular self-dual normal bases) are useful in applications, e.g. the construction of devices for the arithmetic in finite fields (multiplication, exponentiation, discrete logarithms; e.g. see [3, 8, and 18]) and in applications to coding theory, cryptography, and the discrete Fourier transform (see [7, 6, and 4]). Thus it is not surprising that self-dual (normal) bases have found considerable interest in the literature. The first general result is the following theorem of Lempel and Seroussi [13]:

Theorem 1. *E has a trace-orthogonal basis over F. Moreover, E has a self-dual basis over F if and only if either q is even or both q and m are odd.*

In spite of a simplification by Imamura [10], the published proofs of Theorem 1 are lengthy and involved. In §2, we shall give a very simple proof for this result. This will also lead to the following new result (where $\bar{\alpha}$ is called *almost self-dual* if it satisfies (4) and, with possibly one exception, (5)).

Theorem 2. *E has an almost self-dual basis over F.*

In §3, we shall establish a formula for the number of self-dual bases of E over F:

Theorem 3. *The number $sd(m, q)$ of distinct self-dual bases of E over F is*

$$(6) \quad sd(m, q) = \frac{c}{m!} \prod_{i=1}^{m-1} (q^i - a_i),$$

where

$$c = \begin{cases} 0 & \text{if } q \text{ is odd and } m \text{ is even} \\ 1 & \text{if } q \text{ is even} \\ 2 & \text{if } q \text{ and } m \text{ are odd} \end{cases}$$

and where

$$a_i = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Our proof consists of observing that $sd(m, q)$ equals (for $sd(m, q) \neq 0$) the order of the orthogonal group $O(m, q)$ divided by $m!$. The special case of even valued q in Theorem 3 was already observed by Imamura [11] using a direct enumeration.

Recently, a criterion for the existence of a self-dual normal basis of E over F was obtained.

Theorem 4. *E has a self-dual normal basis over F if and only if either m is odd, or q is even and $m \not\equiv 0 \pmod{4}$.*

The necessity of the condition in Theorem 4 follows, for q odd, trivially from Theorem 1; for q even, it is due to Imamura and Morii [12]. The sufficiency of the criterion was recently established by Lempel and Weinberger [14]. Using a connection between self-dual normal bases and orthogonal circulant matrices, we can quote results of MacWilliams [17] to prove the following:

Theorem 5. *Assume that q is a prime, and that either m is odd, or that $q = 2$ and $m \not\equiv 0 \pmod{4}$. Let $f^*(x)$ denote the reciprocal polynomial of $f(x)$. If $(m, q) = 1$ then let $x^m - 1 = (x - 1) \prod_{i=1}^t f_i(x) \prod_{j=t+1}^u g_j(x)$, where $f_i^*(x) = f_i(x)$ and $g_j(x) = h_j(x)h_j^*(x)$, $h_j(x) \neq h_j^*(x)$, and $f_i(x)$, $h_j(x)$, $h_j^*(x)$ are irreducible over F for all $i \in \{1, \dots, t\}$, $j \in \{t+1, \dots, u\}$.*

Let $\deg f_i = 2c_i$ and $\deg h_j = d_j$. Then the number $sdn(m, q)$ of distinct self-dual normal bases of E over F is given by

$$sdn(m, q) = \begin{cases} \frac{2^a}{m} \prod_{i=1}^t (q^{c_i} + 1) \prod_{j=t+1}^u (q^{d_j} - 1) & \text{if } (m, q) = 1 \\ \frac{1}{q} q^{(q-1)(s+b)/2} sdn(s, q) & \text{if } m = sq, \end{cases}$$

where

$$a = \begin{cases} 0 & \text{if } q = 2 \text{ and } m \not\equiv 0 \pmod{4} \\ 1 & \text{if both } q \text{ and } m \text{ are odd,} \end{cases}$$

and

$$b = \begin{cases} 0 & \text{if both } q \text{ and } m \text{ are odd} \\ 1 & \text{if } q = 2 \text{ and } s \text{ is odd.} \end{cases}$$

2. THE EXISTENCE OF SELF-DUAL BASES

In this section, we prove Theorems 1 and 2. We shall use the following well-known lemma; cf. Artin [1]:

Lemma 1. *Let $F = GF(q)$. If q is odd, then there are exactly two equivalence classes of nondegenerate symmetric bilinear forms on F^m , represented by the matrices I and $N = \text{diag}(1, \dots, 1, n)$, where n is an arbitrary nonsquare in F .*

Proof of Theorem 1. We distinguish the cases q even and q odd. First, let q be even; note that then $\text{Tr}(\alpha)^2 = \text{Tr}(\alpha^2)$ for all $\alpha \in E$. Since $\text{Tr}(x) = 1$ describes a hyperplane in the affine geometry $AG(m, q)$, we may select $1 \neq \alpha_1 \in E$ with $\text{Tr}(\alpha_1^2) = 1$. Assume that we have already found $\alpha_1, \dots, \alpha_k \in E$ with $\text{Tr}(\alpha_i \alpha_j) = \delta_{ij}$ for $i, j = 1, \dots, k$, $k < m$, where we also assume that $\alpha_1 + \dots + \alpha_k \neq 1$. It is easily verified that $\alpha_1, \dots, \alpha_k$ are linearly independent over F . We want to select α_{k+1} with $\text{Tr}(\alpha_i \alpha_{k+1}) = \delta_{i, k+1}$ for $i = 1, \dots, k+1$ and $\alpha_1 + \dots + \alpha_{k+1} \neq 1$ (except for the case $k = m-1$ when $\alpha_1 + \dots + \alpha_{k+1} = 1$). First assume that the orthogonal complement $\langle \alpha_1, \dots, \alpha_k \rangle^\perp$ is contained in the hyperplane $H = \{x: \text{Tr}(x) = 0\}$; taking orthogonal complements, it is easily

seen that this implies that $1 \in \langle \alpha_1, \dots, \alpha_k \rangle$. Write $1 = c_1 \alpha_1 + \dots + c_k \alpha_k$; multiplying by α_i and taking the trace shows that $c_i = 1$ for $i = 1, \dots, k$. Thus $\alpha_1 + \dots + \alpha_k = 1$, contradicting the inductive hypothesis. Hence $\langle \alpha_1, \dots, \alpha_k \rangle^\perp$ intersects H in a subspace of dimension $m - k - 1$; thus each coset of H contains q^{m-k-1} elements. In particular, the coset $H_1 = \{x: \text{Tr}(x) = 1\}$ contains an element α_{k+1} with $\text{Tr}(\alpha_{k+1}) = 1$ and $\alpha_1 + \dots + \alpha_k + \alpha_{k+1} \neq 1$, where $k \neq m - 1$. If $k = m - 1$ then $\langle \alpha_1, \dots, \alpha_{m-1} \rangle^\perp \cap H_1 = \{\alpha_m\}$ with $\alpha_1 + \dots + \alpha_m = 1$.

Now assume that q is odd. As noted in the introduction, the trace bilinear form on E over F defined by (2) is a nondegenerate symmetric bilinear form. Note that, in terms of the matrix A defined in (3), the trace bilinear form is represented by the matrix

$$(7) \quad B = A^T A = (\text{Tr}(\alpha_i \alpha_j)),$$

where $\bar{\alpha} = \{\alpha_1, \dots, \alpha_m\}$ is any basis of E over F . We will have a self-dual basis of E over F if and only if the trace bilinear form may be represented by the identity matrix; Lemma 1 shows that this is equivalent to requiring that $\det B$ is a square in F (for any given basis $\bar{\alpha}$). But clearly $\det B = (\det A)^2$ is a square in E , and thus $\det B$ is a square in F if and only if $\det A$ is an element of F , i.e. iff $(\det A)^q = \det A$. Note that $(\det A)^q = \det A^{(q)}$, where $A^{(q)}$ denotes the matrix obtained from A by replacing each entry by its q th power. Thus

$$(\det A)^q = \det A^{(q)} = \det \begin{pmatrix} \alpha_1^q & \dots & \alpha_m^q \\ \alpha_1^{q^2} & \dots & \alpha_m^{q^2} \\ \vdots & \ddots & \vdots \\ \alpha_1^{q^{m-1}} & \dots & \alpha_m^{q^{m-1}} \\ \alpha_1 & \dots & \alpha_m \end{pmatrix},$$

and so $A^{(q)}$ arises from A by a cyclic permutation of the m rows. This shows that

$$(\det A)^q = (-1)^{m-1} \det A,$$

and therefore $\det A \in F$ iff m is odd, proving the theorem. $\square$

We note that for odd q , Lemma 1 always guarantees the existence of a basis $\bar{\alpha}$ for which the trace bilinear form is either represented by I or by N . Such a basis is an almost self-dual basis of E over F , which gives the proof of Theorem 2.

3. ENUMERATION OF SELF-DUAL (NORMAL) BASES

Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be any fixed basis of E over F . Then every basis of E over F may be written in the form $\bar{\beta} = (\beta_1, \dots, \beta_m)$ with

$$(8) \quad \beta_i = \sum_{j=1}^m c_{ij} \alpha_j, \quad (i = 1, \dots, m),$$

where $C = (c_{ij})$ is an invertible $(m \times m)$ -matrix over F . We shall establish two lemmas which are the key to proving Theorems 3 and 5. The first of these is as follows:

Lemma 2. *Assume that $\bar{\alpha}$ is a self-dual basis. Then $\bar{\beta}$ is likewise self-dual if and only if C is an orthogonal matrix (i.e. $CC^T = C^TC = I$).*

Proof. $\bar{\beta}$ is self-dual if and only if for all $i, j = 1, \dots, m$

$$\begin{aligned} \delta_{ij} &= \text{Tr}(\beta_i \beta_j) = \text{Tr} \left(\left(\sum_{h=1}^m c_{ih} \alpha_h \right) \left(\sum_{k=1}^m c_{jk} \alpha_k \right) \right) \\ &= \sum_{h,k=1}^m c_{ih} c_{jk} \text{Tr}(\alpha_h \alpha_k) = \sum_{k=1}^m c_{ik} c_{jk}, \end{aligned}$$

(since $\bar{\alpha}$ is self-dual), which holds if and only if $CC^T = I$. $\square$

Corollary 1. *Denote by $O(m, q)$ the group of orthogonal $m \times m$ -matrices over $GF(q)$. Then $sd(m, q) = (1/m!)|O(m, q)|$, provided that $sd(m, q) \neq 0$.*

Using the well-known formulae for the order of $O(m, q)$ then results in Theorem 3. The required result may be found in the tables of Hirschfeld [9]; an elementary derivation was given by MacWilliams [16]. (We remark that one may similarly count the number of almost self-dual bases for q odd, m even; the result will be $((q-1)/2m!)|O(m, q)|$.) We now prove our second key lemma:

Lemma 3. *Assume that $\bar{\alpha}$ is a normal basis. Then $\bar{\beta}$ is likewise normal if and only if C is a circulant matrix (i.e. $c_{i+1, j+1} = c_{ij}$ for all i and j , where indices are computed modulo m).*

Proof. By hypothesis we may write $\alpha_j = \alpha^{q^j}$ ($j = 1, \dots, m$). Then

$$\beta_i = \sum_{j=1}^m c_{ij} \alpha^{q^j}$$

and therefore

$$\beta_i^q = \sum_{j=1}^m c_{ij} \alpha^{q^{j+1}}.$$

Thus we have $\beta_{i+1} = \beta_i^q$ for $i = 1, \dots, m$ (making $\bar{\beta}$ a normal basis) if and only if

$$\beta_i^q = \sum_{j=1}^m c_{ij} \alpha^{q^{j+1}} = \beta_{i+1} = \sum_{j=1}^m c_{i+1, j} \alpha^{q^j} = \sum_{j=1}^m c_{i+1, j+1} \alpha^{q^{j+1}},$$

for $i = 1, \dots, m$. Clearly this holds if and only if $c_{ij} = c_{i+1, j+1}$ for all $i, j = 1, \dots, m$, i.e. iff C is circulant. $\square$

Corollary 2. *The number of normal bases of E over F is $(1/m)|C(m, q)|$, where $C(m, q)$ denotes the group of invertible circulant $(m \times m)$ -matrices over $GF(q)$.*

Of course, the number of normal bases is well known (see e.g. Lidl and Niederreiter [15] or Berlekamp [2]), and we refrain from restating it. Combining Lemmas 2 and 3, we get our principal result:

Corollary 3. *Assume $sdn(m, q) \neq 0$ (cf. Theorem 4). Then $sdn(m, q) = (1/m)|OC(m, q)|$, where $OC(m, q)$ denotes the group of orthogonal circulant $(m \times m)$ -matrices over $GF(q)$.*

MacWilliams [17] has determined the order of $OC(m, q)$ and described a way of generating the matrices in question, provided that q is prime. Using her results in Corollary 3 then gives Theorem 5.

4. CONCLUSIONS

Self-dual and self-dual normal bases of $GF(q^m)$ over $GF(q)$ are important in a variety of applications. Using only simple techniques from linear algebra and basic facts about finite fields, we have obtained a new short proof for the existence criterion for self-dual bases. We have also enumerated such bases completely, and we enumerated self-dual normal bases if the ground field $GF(q)$ has prime order. Beth and Geiselmann [5] have recently extended MacWilliams' formulae to determine the order of $OC(m, q)$, where q is any prime power, thus completing the enumeration of self-dual normal bases.

REFERENCES

1. E. Artin, *Geometric algebra*, Interscience Publishers, New York, 1957.
2. E. R. Berlekamp, *Algebraic coding theory*, McGraw-Hill, New York, 1968.
3. ———, *Bit serial Reed-Solomon encoders*, IEEE Trans. Inform. Theory **IT-28** (1982), 869–874.
4. T. Beth, *Generalizing the discrete fourier transform*, Discrete Math. **56** (1985), 95–100.
5. T. Beth and W. Geiselmann, *Selbstduale Normalbasen über $GF(q)$* , Archiv. Math. (to appear).
6. W. Diffie and M. E. Hellman, *New directions in cryptography*, IEEE Trans. Inform. Theory **IT-22** (1976), 644–654.
7. W. Fumy, *Orthogonal transform encoding of cyclic codes*, Algebraic Algorithms and Error-Correcting Codes, Lecture Notes in Comput. Sci., vol. 229, Springer-Verlag, 1986, 131–134.
8. W. Geiselmann and D. Gollmann, *Symmetry and duality in normal basis multiplication*, AAECC-6 Proceedings (to appear).
9. J. W. P. Hirschfeld, *Projective geometries over finite fields*, Clarendon Press, Oxford, 1979.
10. K. Imamura, *On self-complementary bases of $GF(q^n)$ over $GF(q)$* , Trans. IECE Japan (Section E) **E66** (1983), 717–721.
11. K. Imamura, *The number of self-complementary bases of a finite field of characteristic two*, IEEE Internat. Sympos. Inform. Theory, Kobe, Japan, 1988.
12. K. Imamura and M. Morii, *Two classes of finite fields which have no self-complementary normal bases*, IEEE Internat. Sympos. Inform. Theory Brighton, England, June 1985.

13. A. Lempel and G. Seroussi, *Factorization of symmetric matrices and trace-orthogonal bases in finite fields*, SIAM J. Comput. **9** (1980), 758–767.
14. A. Lempel and M. J. Weinberger, *Self-complementary normal bases in finite fields*, SIAM J. Disc. Math. **1** (1988), 193–198.
15. R. Lidl and H. Niederreiter, *Finite fields*, Cambridge University Press, 1987.
16. F. J. MacWilliams, *Orthogonal matrices over finite fields*, Amer. Math. Monthly **76** (1969), 152–164.
17. ———, *Orthogonal circulant matrices over finite fields and how to find them*, J. Combin. Theory **10** (1971), 1–17.
18. R. C. Mullin, I. M. Onyszchuk, S. A. Vanstone, and R. M. Wilson, *Optimal normal bases in $GF(p^m)$* , Discrete Appl. Math. **22** (1988–89), 149–161.

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