ON THE EQUATION

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$$

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ABSTRACT. The first eigenvalue $\lambda = \lambda_1$ for the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ is simple in *any* bounded domain. (Through the nonlinear counterpart to the Rayleigh quotient λ_1 is related to the Poincaré inequality.)

1. INTRODUCTION

The first eigenvalue of the operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is here defined as the least real number λ for which the equation

$$(1.1) \quad \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$$

has a nontrivial solution u with zero boundary data in a given bounded domain in the n -dimensional Euclidean space. The first eigenvalue is the minimum of the Rayleigh quotient:

$$(1.2) \quad \lambda = \min_u \frac{\int |\nabla u|^p dx}{\int |u|^p dx}.$$

Here $1 < p < \infty$, and in the linear case $p = 2$ one obtains “the principal frequency”; the associated first eigenfunction u describes the shape of a membrane when it vibrates emitting its gravest tone, cf. [6]. (We shall often use the term *principal frequency* for the nonlinear cases as well.)

1.3. Theorem. *The first eigenvalue is simple in any bounded domain Ω in $\mathbf{R}^n$, i.e., all the associated first eigenfunctions u are merely constant multiples of each other.*

In the classical linear case, i.e., when $p = 2$, this is a well-known result [4, Theorem 8.38, p. 214]. This phenomenon is a reflection of the distinguishing feature of the first eigenfunctions: they never change signs in Ω . Higher eigenvalues are not simple even in the case $\Delta u + \lambda u = 0$, cf. [6, §§5.9B and 7.8].

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Various versions of Theorem 1.3 are known for general exponents p , but, so far as we know, only for a very restricted class of domains. Our contribution is to prove the same result for *any* bounded domain.

F. de Thélin has observed that there exists essentially only one first eigenfunction among the radial functions in a ball [11]. An immediate consequence of the interesting work [8] of Sakaguchi is that the first eigenvalue is simple in convex domains.¹ Theorem 1.3 has been extended to domains of Hölder class $C^{2,\alpha}$ by Anane [1]. We have been informed that Bhattacharya has further extended this to C^2 -domains. This simplicity of the first eigenvalue in C^2 -domains has been proved in a slightly more general setting by Veron and Guedda, cf. [14]. Unfortunately, these proofs rely heavily on deep global regularity results for the gradients of the first eigenfunctions [3], [12]. As this kind of global estimate cannot hold in arbitrary domains, such an approach will artificially restrict the class of domains.

The objective of this note is to present a natural proof for Theorem 1.3. Our method is direct, and no advanced tools are needed. In fact, the proof is based on a few immediate observations and a refined choice of the test-functions in [1].

The analysis can be readily extended to homogeneous eigenvalue problems of the form

$$\operatorname{div} \mathbf{A}(x, \nabla u(x)) + \lambda \rho(x) |u(x)|^{p-2} u(x) = 0$$

where $\mathbf{A}(x, w) \approx |w|^{p-2} w$ fulfills certain restrictive structural conditions and $\rho(x) \geq 0$. The proof is a simple adaptation of the arguments presented here.

Finally, we want to mention that to any bounded domain Ω there is a constant $c(\Omega)$ such that the Poincaré inequality

$$(1.4) \quad \int_{\Omega} |u|^p dx \leq c(\Omega) \int_{\Omega} |\nabla u|^p dx$$

is valid for every $u \in C_0^\infty(\Omega)$ or, more generally, for every $u \in W_0^{1,p}(\Omega)$. The best constant $c(\Omega)$ is the reciprocal of the first eigenvalue in Ω .

Throughout this paper, Ω denotes an arbitrary bounded domain in $\mathbf{R}^n$.

2. THE RAYLEIGH QUOTIENT

Among the equations

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{q-2} u = 0,$$

$1 < p, q < \infty$, only the homogeneous case $q = p$ has the proper structure of a “typical eigenvalue problem”, to quote an expression in [2], where this phenomenon has been studied. In defining the eigenvalues for the operator $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ in a given domain $\Omega \subset \mathbf{R}^n$ we shall interpret equation (1.1) in the weak sense.

¹ Note added in proof. B. Kawohl has kindly pointed out to us that Sakaguchi’s proof is valid in C^2 -domains.

2.1. Definition. We say that λ is an eigenvalue, if there exists a continuous function $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, such that

$$(2.2) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx = \lambda \int_{\Omega} |u|^{p-2} u \eta \, dx$$

whenever $\eta \in W_0^{1,p}(\Omega)$. The function u is called an eigenfunction.

The continuity of u is a redundant requirement in the definition: the weak solutions of (2.2) can be made continuous after a redefinition in a set of measure zero. This is standard elliptic regularity theory. (As a matter of fact, one even has $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$ for some $\alpha > 0$, cf. [12] and [3].)

The least eigenvalue, say $\lambda_1 = \lambda_1(\Omega)$, is obtained as the minimum of the Rayleigh quotients, i.e.,

$$(2.3) \quad \lambda_1 = \inf_v \frac{\int_{\Omega} |\nabla v|^p \, dx}{\int_{\Omega} |v|^p \, dx},$$

the infimum being taken among all $v \in W_0^{1,p}(\Omega)$, $v \not\equiv 0$. Alternatively, one can further restrict the admissible functions to those in $C_0^\infty(\Omega)$. It is easily seen that this minimization problem is equivalent to equation (2.2) with $\lambda = \lambda_1$.

We say that λ_1 is the *first eigenvalue* or the *principal frequency* and the corresponding eigenfunction is called the *first eigenfunction*. In a bounded domain Ω the existence of a first eigenfunction (and of the principal frequency $\lambda_1 > 0$) is established via a minimizing sequence $u_1, u_2, \dots$ for the Rayleigh quotient. By homogeneity the normalization

$$\int_{\Omega} |u_k|^p \, dx = 1 \quad (k = 1, 2, \dots)$$

is possible. In this standard procedure one usually uses the Rellich-Kondrachov imbedding theorem [9, §11, pp. 82–85] and the Radon-Riesz theorem [7, §37, p. 71] related to the uniform convexity of $L^p(\Omega)$. See [10].

If u minimizes (2.3), so does $|u|$. Thus the existence of a first eigenfunction not changing signs in Ω is proved:

2.4. Lemma. *In any bounded domain there is a first eigenfunction $u_{\lambda_1} \geq 0$ corresponding to the principal frequency $\lambda_1 > 0$.*

Furthermore, $u_{\lambda_1} > 0$ if $u_{\lambda_1} \geq 0$. This is a finer point following from the Harnack inequality [13, Theorem 1.1, p. 724].

3. THE FIRST EIGENFUNCTIONS

As before, Ω denotes an arbitrary bounded domain. The crucial part of the proof for Theorem 1.3 is to establish that positive eigenfunctions are essentially unique. The general case can be reduced to this situation. To this end, note that if u is a first eigenfunction, so is $|u|$. By Harnack's inequality [13, Theorem 1.1, p. 724] either $|u| > 0$ in the whole domain or $|u| \equiv 0$, the latter case being excluded for eigenfunctions. By continuity, either u or $-u$ is positive in the whole domain. Hence Theorem 1.3 follows from the following lemma.

3.1. Lemma. Suppose that $u \geq 0$ and $v \geq 0$ are eigenfunctions both corresponding to λ_1 . Then u and v are proportional.

Proof. As Anane has observed in [1], the result would follow by certain balanced calculations, if the function $\eta = u - v^p u^{1-p}$ were, a priori, admissible as test-function in

$$(3.2) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx = \lambda_1 \int_{\Omega} |u|^{p-2} u \eta \, dx$$

and $v - u^p v^{1-p}$ in the similar equation for v . However, this seems to require some regularity of the boundary $\partial\Omega$. For “irregular domains” this is the crucial point.

Therefore we use the modified test-functions

$$(3.3) \quad \eta = \frac{(u + \varepsilon)^p - (v + \varepsilon)^p}{(u + \varepsilon)^{p-1}} \quad \text{and} \quad \frac{(v + \varepsilon)^p - (u + \varepsilon)^p}{(v + \varepsilon)^{p-1}},$$

ε being a positive parameter. Then

$$(3.4) \quad \nabla \eta = \left\{ 1 + (p-1) \left(\frac{v + \varepsilon}{u + \varepsilon} \right)^p \right\} \nabla u - p \left(\frac{v + \varepsilon}{u + \varepsilon} \right)^{p-1} \nabla v,$$

and, by symmetry, the gradient of the test-function in the corresponding equation for v has a similar expression with u and v interchanged. Set

$$u_{\varepsilon} = u + \varepsilon, \quad v_{\varepsilon} = v + \varepsilon.$$

Inserting the chosen test-functions into their respective equations and adding these, we obtain the expression

$$(3.5) \quad \begin{aligned} & \lambda_1 \int_{\Omega} \left[\frac{u^{p-1}}{u_{\varepsilon}^{p-1}} - \frac{v^{p-1}}{v_{\varepsilon}^{p-1}} \right] (u_{\varepsilon}^p - v_{\varepsilon}^p) \, dx \\ &= \int_{\Omega} \left[\left\{ 1 + (p-1) \left(\frac{v_{\varepsilon}}{u_{\varepsilon}} \right)^p \right\} |\nabla u_{\varepsilon}|^p + \left\{ 1 + (p-1) \left(\frac{u_{\varepsilon}}{v_{\varepsilon}} \right)^p \right\} |\nabla v_{\varepsilon}|^p \right] \, dx \\ & \quad - \int_{\Omega} \left[p \left(\frac{v_{\varepsilon}}{u_{\varepsilon}} \right)^{p-1} |\nabla u_{\varepsilon}|^{p-2} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon} + p \left(\frac{u_{\varepsilon}}{v_{\varepsilon}} \right)^{p-1} |\nabla v_{\varepsilon}|^{p-2} \nabla v_{\varepsilon} \cdot \nabla u_{\varepsilon} \right] \, dx \\ &= \int_{\Omega} (u_{\varepsilon}^p - v_{\varepsilon}^p) (|\nabla \log u_{\varepsilon}|^p - |\nabla \log v_{\varepsilon}|^p) \, dx \\ & \quad - \int_{\Omega} p v_{\varepsilon}^p |\nabla \log u_{\varepsilon}|^{p-2} \nabla \log u_{\varepsilon} \cdot (\nabla \log v_{\varepsilon} - \nabla \log u_{\varepsilon}) \, dx \\ & \quad - \int_{\Omega} p u_{\varepsilon}^p |\nabla \log v_{\varepsilon}|^{p-2} \nabla \log v_{\varepsilon} \cdot (\nabla \log u_{\varepsilon} - \nabla \log v_{\varepsilon}) \, dx, \end{aligned}$$

and here the last member is clearly ≤ 0 by inequality (4.1) in the appendix.

It is apparent that

$$(3.6) \quad \lim_{\varepsilon \rightarrow 0+} \int_{\Omega} \left[\frac{u^{p-1}}{u_{\varepsilon}^{p-1}} - \frac{v^{p-1}}{v_{\varepsilon}^{p-1}} \right] (u_{\varepsilon}^p - v_{\varepsilon}^p) \, dx = 0.$$

Let us first consider the case $p \geq 2$. According to inequality (4.3) in the appendix we have

$$(3.7) \quad \begin{aligned} 0 &\leq \frac{1}{2^{p-1}-1} \int_{\Omega} \left(\frac{1}{v_{\varepsilon}^p} + \frac{1}{u_{\varepsilon}^p} \right) |v_{\varepsilon} \nabla u_{\varepsilon} - u_{\varepsilon} \nabla v_{\varepsilon}|^p dx \\ &\leq -\lambda_1 \int_{\Omega} \left[\left(\frac{u}{u_{\varepsilon}} \right)^{p-1} - \left(\frac{v}{v_{\varepsilon}} \right)^{p-1} \right] (u_{\varepsilon}^p - v_{\varepsilon}^p) dx \end{aligned}$$

for every $\varepsilon > 0$. (Inequality (4.3) was used with $w_1 = \nabla \log u_{\varepsilon}$, $w_2 = \nabla \log v_{\varepsilon}$ and vice versa.) Recalling (3.6), letting ε tend toward zero through any positive sequence $\varepsilon_1, \varepsilon_2, \dots$, and using Fatou's lemma in (3.7), we finally arrive at the conclusion that $v \nabla u = u \nabla v$ a.e. in Ω . Hence there is a constant κ such that $u = \kappa v$ a.e. in Ω . By continuity $u = \kappa v$ at every point in Ω . This proves the case $p \geq 2$.

The case $1 < p < 2$ is very similar. Applying inequality (4.4) in the appendix on (3.5) we obtain

$$(3.8) \quad \begin{aligned} 0 &\leq C(p) \int_{\Omega} (u_{\varepsilon} v_{\varepsilon})^p (u_{\varepsilon}^p + v_{\varepsilon}^p) \frac{|v_{\varepsilon} \nabla u_{\varepsilon} - u_{\varepsilon} \nabla v_{\varepsilon}|^2}{(v_{\varepsilon} |\nabla u_{\varepsilon}| + u_{\varepsilon} |\nabla v_{\varepsilon}|)^{2-p}} dx \\ &\leq -\lambda_1 \int_{\Omega} \left[\left(\frac{u}{u_{\varepsilon}} \right)^{p-1} - \left(\frac{v}{v_{\varepsilon}} \right)^{p-1} \right] (u_{\varepsilon}^p - v_{\varepsilon}^p) dx \end{aligned}$$

for every $\varepsilon > 0$. Using (3.6), we again arrive at the desired dependence $u = \kappa v$ for some constant. This concludes the proof. $\square$

Remarks. (1°) If $v \geq 0$ is any eigenfunction in Ω corresponding to the eigenvalue λ , then $\lambda = \lambda_1$, i.e., *only the first eigenfunctions are positive*. See [1]. Indeed, if $u \geq 0$ denotes a first eigenfunction, then the same procedure as above yields

$$\int_{\Omega} \left[\lambda_1 \left(\frac{u}{u_{\varepsilon}} \right)^{p-1} - \lambda \left(\frac{v}{v_{\varepsilon}} \right)^{p-1} \right] (u_{\varepsilon}^p - v_{\varepsilon}^p) dx \leq 0$$

and arguing as before, we arrive at

$$(\lambda_1 - \lambda) \int_{\Omega} (u^p - v^p) dx \leq 0.$$

This leads to a contradiction, if $\lambda > \lambda_1$, since v can be replaced by any of the functions $2v, 3v, 4v, \dots$. Thus $\lambda = \lambda_1$.

(2°) The first eigenvalue is isolated. This was proved in [1] for sufficiently regular domains, but those proofs can be carried over to general domains. The necessary modifications are suggested by the constructions in the proof for Lemma 3.1.

4. APPENDIX: AN INEQUALITY

The familiar inequality

$$(4.1) \quad |w_2|^p > |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1)$$

for points in $\mathbf{R}^n$, $w_1 \neq w_2$, $p > 1$, is just a restating of the strict convexity of the mapping $w \rightarrow |w|^p$. It is sometimes convenient to express this strictness in the following quantitative way:

4.2. Lemma. *If $p \geq 2$, then*

$$(4.3) \quad |w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + \frac{|w_2 - w_1|^p}{2^{p-1} - 1}$$

for all points w_1 and w_2 in $\mathbf{R}^n$.

If $1 < p < 2$, then

$$(4.4) \quad |w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + C(p) \frac{|w_2 - w_1|^2}{(|w_1| + |w_2|)^{2-p}}$$

for all points w_1 and w_2 in $\mathbf{R}^n$. Here $C(p)$ is a positive constant depending only on p .

Observe that

$$(4.5) \quad \frac{|w_2 - w_1|^2}{(|w_1| + |w_2|)^{2-p}} \leq |w_2 - w_1|^p \quad (1 \leq p \leq 2).$$

Since we have not been able to find any reference for these natural inequalities, we have included a proof.

Proof. For $p \geq 2$ inequality (4.3) follows from Clarkson's inequality

$$(4.6) \quad |w_1|^p + |w_2|^p \geq 2 \left| \frac{w_1 + w_2}{2} \right|^p + 2 \left| \frac{w_1 - w_2}{2} \right|^p.$$

To see this, use (4.1) to get

$$(4.7) \quad \left| \frac{w_1 + w_2}{2} \right|^p \geq |w_1|^p + \frac{1}{2}p|w_1|^{p-2}w_1 \cdot (w_2 - w_1).$$

Combining the two inequalities we arrive at

$$(4.8) \quad |w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + 2 \left| \frac{w_1 - w_2}{2} \right|^p.$$

This is the desired inequality but with the constant 2^{1-p} in place of $\frac{1}{2^{p-1}-1}$. Repeating this procedure, starting again with (4.6) but now using (4.8) instead of (4.1), we get the constant improved to $2^{1-p} + 4^{1-p}$. By iteration one finally obtains the constant

$$2^{1-p} + 4^{1-p} + 8^{1-p} + \dots = \frac{1}{2^{p-1} - 1}$$

in (4.3). (The best constant is irrelevant for our purpose here.)

Let us now consider the case $1 < p < 2$. In order to derive (4.4) we fix w_1 and w_2 . Expanding the function

$$f(t) = |w_1 + t(w_2 - w_1)|^p$$

according to Maclaurin's formula

$$f(1) = f(0) + f'(0) + \int_0^1 (1-t)f''(t) dt$$

we have

$$(4.9) \quad |w_2|^p = |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + \int_0^1 (1-t)f''(t) dt$$

provided that $|w_1 + t(w_2 - w_1)| \neq 0$, when $0 \leq t \leq 1$. But the case when $w_1 + t(w_2 - w_1) = 0$ for some t , $0 \leq t \leq 1$, is easily checked. By direct calculation

$$\begin{aligned} f''(t) &= p(p-2)|w_1 + t(w_2 - w_1)|^{p-4} \{(w_1 + t(w_2 - w_1)) \cdot (w_2 - w_1)\}^2 \\ &\quad + p|w_1 + t(w_2 - w_1)|^{p-2}|w_2 - w_1|^2, \end{aligned}$$

and the Schwarz inequality yields

$$(4.10) \quad f''(t) \geq p(p-1)|w_1 + t(w_2 - w_1)|^{p-2}|w_2 - w_1|^2.$$

Returning to (4.9), we have

$$\int_0^1 (1-t)f''(t) dt \geq \frac{3}{4} \int_0^{1/4} f''(t) dt$$

and, since $|w_1| + |w_2| \geq |w_1 + t(w_2 - w_1)|$ we finally arrive at (4.4) with $C(p) = 3p(p-1)4^{-2}$. Here (4.10) was used. $\square$

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