ON THE CHARACTERIZATION OF $H^p(R^n)$ IN TERMS OF FOURIER MULTIPLIERS

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ABSTRACT. Let $p \in (0, 1]$, let $f \in L^2(R^n) \cap H^p(R^n)$ and let $\theta \in C(R^n \setminus \{0\})$ be homogeneous of degree zero. We will give one sufficient condition in order for f and θ to satisfy

$$\|f\|_{H^p} \leq C\{\|f\|_{L^p} + \|\mathcal{F}^{-1}\{\theta(\xi)\mathcal{F}f(\xi)\}\|_{L^p}\}.$$

Functions considered in this note are complex valued. Let $\Phi \in \mathcal{S}(R^n)$ and $\int_{R^n} \Phi(x) dx \neq 0$. For a tempered distribution $f \in \mathcal{S}(R^n)'$, and $x = (x_1, \dots, x_n) \in R^n$ let

$$f_{\Phi}^+(x) = \sup_{t>0} |f * \Phi_t(x)|,$$

where $\Phi_t(y) = t^{-n}\Phi(y/t)$ and where $*$ denotes the convolution on R^n . Freezing this Φ , for $p \in (0, 1]$ let

$$\|f\|_{H^p} = \|f\|_{H^p(R^n)} = \|f_{\Phi}^+\|_{L^p(R^n)}.$$

It was shown by [FS] that $\|\cdot\|_{H^p}$ is essentially independent of the choice of Φ . Let

$$H^p(R^n) = \{f \in \mathcal{S}(R^n)': \|f\|_{H^p} < \infty\}.$$

For $\theta \in L^\infty(R^n)$ and for $f \in L^2(R^n)$ let

$$\mathcal{M}_{\theta}f = \mathcal{F}^{-1}(\theta(\xi)\mathcal{F}f(\xi)),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. When θ satisfies

$$\theta(r\xi) \equiv \theta(\xi) \quad \text{for any } \xi \in R^n \setminus \{0\} \quad \text{and for any } r > 0,$$

θ is said to be homogeneous of degree zero.

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We prepare some notation. Let $n \in \{2, 3, 4, \dots\}$. For $\xi = (\xi_1, \dots, \xi_n) \in R^n$ and for $\varepsilon > 0$ let

$$\begin{aligned} |\xi| &= \{\xi_1^2 + \dots + \xi_n^2\}^{1/2}, \\ \mathcal{R} &= \{\xi \in R^n : 1 < |\xi| < 2\}, \\ \xi' &= (\xi_2, \dots, \xi_n) \in R^{n-1}, \\ |\xi'| &= \{\xi_2^2 + \dots + \xi_n^2\}^{1/2}, \\ 0' &= (0, \dots, 0) \in R^{n-1}, \\ \Gamma_\varepsilon &= \{\xi \in R^n : |\xi'| \leq \varepsilon |\xi_1|\}, \\ \Gamma_\varepsilon^+ &= \{\xi \in \Gamma_\varepsilon : \xi_1 \geq 0\}, \\ \Gamma_\varepsilon^- &= \{\xi \in \Gamma_\varepsilon : \xi_1 \leq 0\}, \\ \Gamma_\varepsilon^0 &= \text{the interior of } \Gamma_\varepsilon. \end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ let

$$\begin{aligned} |\alpha| &= \alpha_1 + \dots + \alpha_n, \\ \alpha' &= (\alpha_2, \dots, \alpha_n), \\ D_x^\alpha &= \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}. \end{aligned}$$

For a set E , χ_E denotes its characteristic function. The letter C denotes various positive constants. For example, the notation $C_1(p)$ means that C_1 is a positive constant depending only on p .

Our result is the following.

Theorem. *Let*

- (1) $n \in \{2, 3, 4, \dots\},$
- (2) $p \in (0, 1],$
- (3) $d(p, n) = [n(1/p - \frac{1}{2})] + 1,$
- (4) $\theta \in C^{d(p, n)+1}(\Gamma_1^0),$
- (5) $\theta(r\xi) = \theta(\xi) \quad \text{for any } \xi \in \Gamma_1^0 \quad \text{and any } r > 0,$
- (6) $\theta(1, 0') \neq \theta(-1, 0').$

Then there exist $\varepsilon_0(\theta, p) \in (0, 1)$ and $C_0(\theta, p) < \infty$ such that if $f \in L^2(R^n) \cap H^p(R^n)$ and if

$$(7) \quad \text{supp } \mathcal{F} f \subset \Gamma_{\varepsilon_0},$$

then

$$(8) \quad \|f\|_{H^p} \leq C_0 \{\|f\|_{L^p} + \|\mathcal{M}_\theta f\|_{L^p}\},$$

where $[a]$ denotes the integral part of a real number a and where ε_0 and C_0 depend only on p , n , $|\theta(1, 0')|$, $|\theta(1, 0') - \theta(-1, 0')|$ and

$$(9) \quad \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_\xi^\alpha \theta\|_{L^\infty(\mathcal{R} \cap \Gamma_1^0)}.$$

Remark. In (4), (5), and (9) we can replace Γ_1^0 by Γ_δ^0 , where $\delta > 0$, by making ε_0 and C_0 dependent on δ .

Lemma A. Let $f \in L^2(R^1)$ and $p \in (0, 1]$. Then

$$\|f\|_{H^p(R^1)} \leq C_1(p) \{ \|f\|_{L^p(R^1)} + \|Hf\|_{L^p(R^1)} \}$$

where

$$Hf = \mathcal{F}^{-1} \{ -i \operatorname{sign}(s) \mathcal{F} f(s) \}.$$

This follows from [FS] Theorem 11 and from

$$\begin{aligned} & \sup_{t>0} \int_{-\infty}^{+\infty} \{ |u_0(s, t)|^2 + |u_1(s, t)|^2 \}^{p/2} ds \\ &= \int_{-\infty}^{+\infty} \{ |f(s)|^2 + |Hf(s)|^2 \}^{p/2} ds \end{aligned}$$

where

$$(u_0(s, t), u_1(s, t)) = \int_{-\infty}^{+\infty} (f(r), Hf(r)) \pi^{-1} t ((s-r)^2 + t^2)^{-1} dr.$$

Lemma B. Assume (1) and (2). Let $\varepsilon \in (0, +\infty)$ and

$$(10) \quad f \in L^2(R^n) \quad \text{and} \quad \operatorname{supp} \mathcal{F} f \subset \Gamma_\varepsilon.$$

Then

$$\|f\|_{H^p}^p \leq C_2(\varepsilon, p, n) \int_{R^{n-1}} \|f(\cdot, x')\|_{H^p(R^1)}^p dx' \leq C_2^2 \|f\|_{H^p}^p,$$

where $\|f(\cdot, x')\|_{H^p(R^1)}$ denotes the $H^p(R^1)$ -norm of $f(x_1, x')$ as a function of $x_1 \in R^1$ with $x' \in R^{n-1}$ fixed.

This is due to [CD, p. 233]. Since this lemma plays a pivotal role in our argument, we give its proof.

Proof. Let $\phi \in \mathcal{S}(R^1)$ be such that

$$(\mathcal{F}\phi)(s) = \begin{cases} 1 & \text{if } s \in [-1, 1] \\ 0 & \text{if } s \in (-2, 2)^c. \end{cases}$$

Let $\Phi \in \mathcal{S}(R^n)$ be such that

$$\mathcal{F}\Phi(\xi) = \mathcal{F}\Phi(\xi_1, \xi_2, \dots, \xi_n) = \mathcal{F}\phi(\xi_1) \mathcal{F}\phi(\xi_2/(2\varepsilon)) \cdots \mathcal{F}\phi(\xi_n/(2\varepsilon)).$$

Then for f in (10) we have

$$(11) \quad \int_{R^n} f(x-y) \Phi_t(y) dy = \int_{R^1} f(x_1 - y_1, x') \frac{1}{t} \phi\left(\frac{y_1}{t}\right) dy_1$$

for a.e. $(x, t) \in R_+^{n+1}$. Since the left-hand side of (11) is continuous as a function of $(x, t) \in R^n \times (0, +\infty)$ and since the right-hand side of (11) is continuous as a function of $(x_1, t) \in R_+^2$ for a.e. $x' \in R^{n-1}$, we have

$$f_{\Phi}^+(x_1, x') = \sup_{t>0} \left| \int_{R^1} f(x_1 - y_1, x') \frac{1}{t} \phi\left(\frac{y_1}{t}\right) dy_1 \right|$$

for all $x_1 \in R^1$ for a.e. $x' \in R^{n-1}$. Thus

$$\int f_{\Phi}^+(x)^p dx = \int_{R^{n-1}} dx' \int_{R^1} \sup_{t>0} \left| \int_{R^1} f(x_1 - y_1, x') \frac{1}{t} \phi\left(\frac{y_1}{t}\right) dy_1 \right|^p dx_1$$

which implies the desired result. $\square$

Lemma C. Assume (1) and (2). Let $\varepsilon \in (0, +\infty)$. Let $\theta_0 \in L^\infty(R^n)$,

$$\theta_0(\xi) \equiv a \quad \text{on } \Gamma_\varepsilon^+, \quad \theta_0(\xi) \equiv b \quad \text{on } \Gamma_\varepsilon^- \\ a \neq b.$$

Assume (10). Then

$$\|f\|_{H^p} \leq C_3(|a|, |a-b|, \varepsilon, p, n) \{ \|f\|_{L^p} + \|\mathcal{M}_{\theta_0} f\|_{L^p} \}.$$

Since we may assume “ $a = 1$, $b = -1$ ”, Lemma C is an easy consequence of the combination of Lemmas B and A. (See also [CD, p. 231], Theorem I.)

Lemma D. Assume (2). Let $\mu \in C^{d(p,n)}(R^n \setminus \{0\})$ be homogeneous of degree zero. Let $f \in L^2(R^n)$. Then

$$\|\mathcal{M}_\mu f\|_{H^p} \leq C_4(p, n) \left\{ \sum_{|\alpha| \leq d(p,n)} \|D_\xi^\alpha \mu\|_{L^2(\mathcal{R})} \right\} \|f\|_{H^p}.$$

This is due to [CT] Theorem 4.6 and [TW] Theorem 4.2.

Lemma 1. Assume (1) and (2). Let $\varepsilon \in (0, 1]$. Assume (10). Let

$$(12) \quad \tilde{f}(x_1, x') = \varepsilon^{-n+1} f(x_1, x'/\varepsilon) (= \mathcal{F}^{-1} \{ \mathcal{F} f(\xi_1, \varepsilon \xi') \}).$$

Then

$$\|f\|_{H^p} \leq C_5(p, n) \varepsilon^{(n-1)(1-1/p)} \|\tilde{f}\|_{H^p} \leq C_5^2 \|f\|_{H^p}.$$

Proof. Since

$$\text{supp } \mathcal{F} \tilde{f}, \text{ supp } \mathcal{F} f \subset \Gamma_1,$$

we get

$$\begin{aligned} & \varepsilon^{(n-1)(p-1)} \|\tilde{f}\|_{H^p}^p \\ & \approx \varepsilon^{(n-1)(p-1)} \int_{R^{n-1}} \|\tilde{f}(\cdot, x')\|_{H^p(R^1)}^p dx', \quad \text{by Lemma B} \\ & \quad \text{and by } \text{supp } \mathcal{F} \tilde{f} \subset \Gamma_1 \\ & = \varepsilon^{-n+1} \int_{R^{n-1}} \|f(\cdot, x'/\varepsilon)\|_{H^p(R^1)}^p dx' = \int_{R^{n-1}} \|f(\cdot, x')\|_{H^p(R^1)}^p dx' \\ & \approx \|f\|_{H^p}^p \quad \text{by Lemma B and by } \text{supp } \mathcal{F} f \subset \Gamma_1. \quad \square \end{aligned}$$

Lemma 2. Assume (1) and (2). Let $\varepsilon \in (0, \frac{1}{2}]$. Let

$$(13) \quad \nu \in C^{d(p,n)+1}(\Gamma_{2\varepsilon}^0)$$

$$\nu(r\xi) = \nu(\xi) \quad \text{for any } \xi \in \Gamma_{2\varepsilon}^0 \quad \text{and any } r > 0,$$

$$(14) \quad \nu(1, 0') = \nu(-1, 0') = 0.$$

Assume (10). Then

$$\|\mathcal{M}_\nu f\|_{H^p} \leq C_6(p, n)\varepsilon \left\{ \sum_{1 \leq |\alpha| \leq d(p,n)+1} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \right\} \|f\|_{H^p}.$$

Proof. Let

$$(15) \quad \nu^*(\xi) = \nu(\xi_1, \varepsilon\xi').$$

Note that

$$(16) \quad D_{\xi_1}^{\alpha_1} \nu^*(\xi_1, 0') \equiv 0 \quad \text{for } \alpha_1 = 0, 1, 2, \dots$$

$$(17) \quad D_\xi^\alpha \nu^*(\xi) = \varepsilon^{|\alpha'|} (D_\xi^\alpha \nu)(\xi_1, \varepsilon\xi').$$

So, if $\xi \in \Gamma_2^0 \cap \mathcal{R}$ and if $\alpha_1 \in \{0, 1, 2, \dots, d(p, n)\}$, then by (16), (17), and by the mean value theorem we have

$$(18) \quad |D_{\xi_1}^{\alpha_1} \nu^*(\xi)| = |D_{\xi_1}^{\alpha_1} \nu(\xi_1, \varepsilon\xi')|$$

$$\leq \sum_{|\alpha'|=1} \|D_\xi^{(\alpha_1, \alpha')} \nu\|_{L^\infty(\mathcal{R}_1 \cap \Gamma_{2\varepsilon}^0)} |\varepsilon\xi'|$$

$$\leq 2\varepsilon \sum_{1 \leq |\alpha| \leq d(p,n)+1} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R}_1 \cap \Gamma_{2\varepsilon}^0)}$$

where

$$\mathcal{R}_1 = \{\xi \in R^n : 5^{-1/2} < |\xi| < 2\}.$$

If $\xi \in \mathcal{R} \cap \Gamma_2^0$, if $|\alpha| \leq d(p, n)$ and if $|\alpha'| \geq 1$, then

$$(19) \quad |D_\xi^\alpha \nu^*(\xi)| \leq \varepsilon \sup_{\xi \in \mathcal{R} \cap \Gamma_2^0} |(D_\xi^\alpha \nu)(\xi_1, \varepsilon\xi')| \quad \text{by (17)}$$

$$\leq \varepsilon \sum_{1 \leq |\alpha| \leq d(p,n)} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R}_1 \cap \Gamma_{2\varepsilon}^0)}.$$

Take $\psi \in C^\infty(0, +\infty)$ such that

$$\psi(s) = \begin{cases} 1 & \text{on } (0, 1], \\ 0 & \text{on } [\frac{3}{2}, +\infty). \end{cases}$$

Put

$$\mu(\xi) = \nu^*(\xi) \psi(|\xi'|/|\xi_1|).$$

Then

$$(20) \quad \mu = \nu^* \text{ on } \Gamma_1$$

and

$$\begin{aligned}
 (21) \quad & \sum_{|\alpha| \leq d(p, n)} \|D_\xi^\alpha \mu\|_{L^2(\mathcal{R})} \\
 & \leq C(\psi, p, n) \varepsilon \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R}_1 \cap \Gamma_{2\varepsilon}^0)} \quad \text{by (18)–(19)} \\
 & \leq C_7(\psi, p, n) \varepsilon \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \quad \text{by (13).}
 \end{aligned}$$

Define $\tilde{f}$ by (12). Similarly, let

$$(\mathcal{M}_\nu f)^\sim = \mathcal{F}^{-1} \{ \nu(\xi_1, \varepsilon \xi') \mathcal{F} f(\xi_1, \varepsilon \xi') \} = \mathcal{F}^{-1} \{ \nu^*(\xi) \mathcal{F} \tilde{f}(\xi) \} = \mathcal{M}_\nu^* \tilde{f}.$$

Then

$$\begin{aligned}
 \|\mathcal{M}_\nu f\|_{H^p} & \leq C_5 \varepsilon^{(n-1)(1-1/p)} \|(\mathcal{M}_\nu f)^\sim\|_{H^p} \\
 & \quad \text{by Lemma 1 and by } \text{supp } \mathcal{F}(\mathcal{M}_\nu f) \subset \Gamma_\varepsilon \\
 & = C_5 \varepsilon^{(n-1)(1-1/p)} \|\mathcal{M}_\nu^* \tilde{f}\|_{H^p} \\
 & = C_5 \varepsilon^{(n-1)(1-1/p)} \|\mathcal{M}_\mu \tilde{f}\|_{H^p} \quad \text{by (20) and by } \text{supp } \mathcal{F} \tilde{f} \subset \Gamma_1 \\
 & \leq C_5 \varepsilon^{(n-1)(1-1/p)} C_4 C_7 \varepsilon \left\{ \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \right\} \|\tilde{f}\|_{H^p} \\
 & \quad \text{by Lemma D and (21)} \\
 & \leq C_5^2 C_4 C_7 \varepsilon \left\{ \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_\xi^\alpha \nu\|_{L^\infty(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \right\} \|f\|_{H^p} \\
 & \quad \text{by Lemma 1 and (10). } \square
 \end{aligned}$$

Proof of theorem. Let

$$\begin{aligned}
 \theta_0(\xi) & = \begin{cases} \theta(1, 0') & \text{if } \xi_1 > 0, \\ \theta(-1, 0') & \text{if } \xi_1 < 0, \end{cases} \\
 & \quad \varepsilon \in (0, \tfrac{1}{2}], \\
 & \quad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n), \\
 & \quad \text{supp } \mathcal{F} f \subset \Gamma_\varepsilon.
 \end{aligned}$$

Then Lemma C implies

$$(22) \quad \|f\|_{H^p} \leq C_8 \{ \|f\|_{L^p} + \|\mathcal{M}_{\theta_0} f\|_{L^p} \}$$

where

$$C_8 = C_3(|\theta(1, 0')|, |\theta(1, 0') - \theta(-1, 0')|, \tfrac{1}{2}, p, n).$$

Let

$$\nu(\xi) = \theta(\xi) - \theta_0(\xi).$$

Then

$$\begin{aligned} \|\mathcal{M}_{\theta_0} f\|_{L^p} &= \|\mathcal{M}_{\theta} f - \mathcal{M}_{\nu} f\|_{L^p} \leq 2^{1/p} \|\mathcal{M}_{\theta} f\|_{L^p} + 2^{1/p} \|\mathcal{M}_{\nu} f\|_{L^p} \\ &\leq 2^{1/p} \|\mathcal{M}_{\theta} f\|_{L^p} + 2^{1/p} C_6 \varepsilon \left\{ \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_{\xi}^{\alpha} \nu\|_{L^{\infty}(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \right\} \|f\|_{H^p} \end{aligned}$$

by Lemma 2

$$= 2^{1/p} \|\mathcal{M}_{\theta} f\|_{L^p} + 2^{1/p} C_6 \varepsilon \left\{ \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_{\xi}^{\alpha} \theta\|_{L^{\infty}(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \right\} \|f\|_{H^p}.$$

Therefore, substituting this into (22) gives

$$\begin{aligned} \|f\|_{H^p} &\leq C_8 \|f\|_{L^p} + C_8 2^{1/p} \|\mathcal{M}_{\theta} f\|_{L^p} \\ &\quad + C_8 2^{1/p} C_6 \varepsilon \left\{ \sum_{1 \leq |\alpha| \leq d(p, n)+1} \|D_{\xi}^{\alpha} \theta\|_{L^{\infty}(\mathcal{R} \cap \Gamma_{2\varepsilon}^0)} \right\} \|f\|_{H^p}. \end{aligned}$$

Thus, taking ε small enough, we get our theorem. $\square$

Added in proof. I did not mention T. Wolff's recent (unpublished?) work on harmonic gradients in R^3 , because it has no relation to my result of this paper. But, for the sake of complete fairness I write here that I know T. Wolff's recent remarkable work on harmonic gradients in R^3 .

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