LOCALLY EFFICIENT MONOTONE OPERATORS

ANDREI VERONA AND MARIA ELENA VERONA

(Communicated by William J. Davis)

ABSTRACT. We study monotone operators on quasi open or convex subsets of a real Banach space X (quasi open means that the contingent cone at each point equals X). Among others we characterize the maximality of such an operator in terms of its w^* -upper semicontinuity properties and, in the case of a convex domain, also in terms of its behavior at the support points. We next give sufficient conditions for such an operator to be generically single valued, extending Kenderov's theorems. As an application we reobtain generic Gâteaux and Fréchet differentiability results for convex functions defined on not necessarily open convex sets.

1. INTRODUCTION

Let X be a real Banach space. We shall denote by X^* its topological dual, by $B(x, \varepsilon)$ the ε -ball in X centered at x and by B^* the closed unit ball in X^* (centered at the origin).

Let A be a subset of X . For $x \in A$ let $K_x(A)$ be the set of all $y \in X$ such that there exist a sequence (t_n) of positive real numbers convergent to 0 and a sequence $(y_n) \subset X$ convergent to y , with $x + t_n y_n \in A$. $K_x(A)$ is called the *contingent cone* to A at x . It is well known that $K_x(A)$ is a closed cone and that $K_x(A) = K_x(\text{cl}(A))$ (cl stands for the topological closure). If $C \subseteq X$ is convex and $x \in C$ then $K_x(C) = \text{cl}(C_x)$, where $C_x = \{y \in X : \text{there exists } t > 0 \text{ such that } x + ty \in C\}$ is the cone generated by C from x . We shall denote by C_x^* the normal cone to C at x , i.e. $C_x^* = \{x^* \in X^* : x^*(x - y) \geq 0 \text{ for any } y \in C\} = \{x^* \in X^* : x^*(z) \leq 0 \text{ for any } z \in C_x\}$.

Definition. A point $x \in A$ is called a *quasi interior point* of A if $K_x(A) = X$; the set of all quasi interior points of A is called the *quasi interior* of A and is denoted $\text{qi}(A)$. A is called *quasi open* if it is equal to its quasi interior.

Obviously any open set is quasi open. Other important examples of quasi open sets are those of the form $N(C)$, where C is a convex set in X and $N(C)$ denotes the set of all nonsupport points of C . Finally, any dense (respectively relatively open) subset of a quasi open set is quasi open.

Received by the editors February 12, 1988 and, in revised form, September 6, 1989.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 47H05, 49A51, 58C20.

©1990 American Mathematical Society
0002-9939/90 \$1.00 + \$.25 per page

Let us now recall some old definitions and introduce some new ones concerning monotone operators.

Definitions. Let A be a subset of X .

(1) A subset $G \subseteq A \times X^*$ is called *monotone* if

$$\langle x^* - y^*, x - y \rangle \geq 0 \text{ for any } (x, x^*), (y, y^*) \in G.$$

If in addition G is maximal (under set inclusion) among all monotone sets contained in $A \times X^*$, then G is called *maximal monotone* in $A \times X^*$.

(2) A multivalued map $T: A \rightarrow 2^{X^*}$, with $T(x) \neq \emptyset$ for all $x \in A$, is called a (*maximal*) *monotone operator* on A if its graph $G(T)$ is a (maximal) monotone set in $A \times X^*$.

(3) Given a multivalued map $F: A \rightarrow 2^{X^*}$ and $r > 0$ let $A_r = \{x \in A : F(x) \cap rB^* \neq \emptyset\}$ and define $F_r: A_r \rightarrow 2^{X^*}$ by $F_r(x) = F(x) \cap rB^*$. F is called *r-efficient* at $a \in A$ if $F_r(x) \neq \emptyset$ for all x in a relative neighborhood of a in A . If F is *r-efficient* at a but we do not need to specify r , we shall say simply that F is *efficient* at a . If for every $a \in A$ there exists $r_a > 0$ such that F is r_a -efficient at a , we call F *locally efficient*.

(4) A multivalued map $F: A \rightarrow 2^{X^*}$ is called *convex w^* -usco* if $F(x)$ is a nonempty, compact and convex set in X^* for each $x \in A$ and F is norm-to- w^* upper semicontinuous; if in addition its graph does not properly contain the graph of any other convex w^* -usco map on A , F is called *minimal*.

Our first main result, stated below, provides a characterization of the maximal monotone operators among the locally efficient monotone operators on convex sets: a locally efficient monotone operator on a convex set C is maximal monotone if and only if it is convex and w^* -closed valued, w^* -upper semicontinuous in a certain sense and, for every $x \in C$, $T(x)$ is invariant under translations by elements in the normal cone to C at x . More precisely

Theorem A. Let $C \subseteq X$ be convex, $A \subseteq C$ be such that $K_x(A) = K_x(C)$ for each $x \in A$ and let $T: A \rightarrow 2^{X^*}$ be a locally efficient monotone operator. For every $x \in A$ choose $r_x > 0$ such that T is r_x -efficient at x . The following assertions are equivalent:

- (1) T is a maximal monotone operator on A ;
- (2) T is convex and w^* -closed valued, T_r is w^* -usco for every $r > 0$ for which $A_r \neq \emptyset$ and $T(x) + C_x^* \subseteq T(x)$ for every $x \in A$;
- (3) T is convex and w^* -closed valued, T_{r_x} is w^* -usco at x and $T(x) + C_x^* \subseteq T(x)$ for every $x \in A$.

As a consequence we obtain the next corollary which generalizes the well-known fact that a monotone operator on an open set is maximal monotone if and only if it is w^* -usco.

Corollary I. Let $A \subseteq X$ be quasi open and $T: A \rightarrow 2^{X^*}$ be a locally efficient monotone operator. Then T is maximal monotone if and only if it is convex w^* -usco.

The following result shows that monotone operators on certain domains have unique maximal monotone extensions (over the same domain). As above, in the case of open domains, this was known (see for example [5], Theorem 7.13).

Corollary II. *Let $C \subseteq X$ be convex, $A \subseteq C$ be such that $K_x(A) = K_x(C)$ for each $x \in A$ and let $T: A \rightarrow 2^{X^*}$ be a locally efficient monotone operator. Then there exists a unique maximal monotone operator $M: A \rightarrow 2^{X^*}$ such that $T(x) \subseteq M(x)$ for every $x \in A$. If A is quasi open on X then M can be described as follows: let $\hat{T}$ be the monotone operator on A whose graph is the closure (in $A \times X^*$) of the graph of T ; then $M(x)$ is the w^* -closed convex hull of $\hat{T}(x)$.*

The next result may also be useful.

Corollary III. *Let $T: A \rightarrow 2^{X^*}$ be a locally efficient maximal monotone operator and $A_0 \subseteq A$ be such that $K_x(A_0) = K_x(A)$ for every $x \in A_0$. Assume that A is either convex or quasi open. Then the restriction of T to A_0 is maximal monotone (on A_0).*

Theorem B. *Let A be a quasi open, Baire subset of the Banach space X and let $T: A \rightarrow 2^{X^*}$ be a maximal monotone operator. Then there exists a dense G_δ subset D of A such that*

- (1) $T|D$ is a minimal convex w^* -usco map;
- (2) for each $x \in D$, $T(x)$ lies in a sphere of X^* and T is norm-to- w^* -usco at x .

As a consequence we obtain the next corollary.

Corollary. *Let A be a quasi open Baire subset of the Banach space X and let $T: A \rightarrow 2^{X^*}$ be a maximal monotone operator. Assume that X is Asplund (resp. in Stegall's class $\mathcal{S}$). Then there exists a dense G_δ subset A_0 of A such that T is single valued and norm-to-norm (resp. norm-to- w^*) upper semicontinuous at each point of A_0 .*

Note. For an open A , the second assertion in Theorem B is due to Kenderov ([2]). He also proved the assertions in the last corollary in the following context: A open and X Asplund (resp. X has an equivalent norm whose dual norm is strictly convex)(see [3] and [2]).

2. LOCALLY EFFICIENT MONOTONE OPERATORS

In this section we shall extend to our context some standard results in the theory of monotone operators, prove some new ones and use them to prove Theorems A and B.

Lemma 2.1. *Let A be a quasi open subset of X , $T: A \rightarrow 2^{X^*}$ be a monotone operator and let $a \in A$. Then T is locally bounded at a if and only if it is efficient at a .*

Proof. If T is locally bounded at a , it is obviously efficient at a . Assume it is r -efficient at $a \in A$. Then there exists $\varepsilon > 0$ such that $T(x) \cap rB^* \neq \emptyset$ for every $x \in B(a, \varepsilon) \cap A$. Let $x \in B(a, \varepsilon) \cap A$ and $y \in X$; since $x \in \text{qi}(A)$ there exist a sequence (t_n) of positive real numbers convergent to 0 and a sequence $(y_n) \subset X$ convergent to y such that $x + t_n y_n \in A \cap B(x, \varepsilon)$. It follows that for each n there exists $y_n^* \in T(x + t_n y_n) \cap rB^*$. Since T is monotone, for every $x^* \in T(x)$ we have $\langle y_n^* - x^*, t_n y_n \rangle \geq 0$ which implies that

$$x^*(y_n) \leq y_n^*(y_n) \leq r\|y_n\|$$

and thus $x^*(y) \leq r\|y\|$. This shows that $\|x^*\| \leq r$ and therefore the lemma is proved.

Remark. A monotone operator is always locally bounded on the interior of its domain (see [7]; another proof, in a somewhat more general context, can be found in [1]).

Since we shall use it several times, let us recall the following well-known lemma.

Lemma 2.2. *Let $A \subseteq X$ and $T: A \rightarrow 2^{X^*}$ be a monotone operator.*

(1) *T is maximal if and only if $(x, x^*) \in A \times X^*$ and $\langle x^* - y^*, x - y \rangle \geq 0$ for all $(y, y^*) \in G(T)$ imply $x^* \in T(x)$.*

(2) *If T is maximal, then $T(x)$ is a convex w^* -closed subset of X^* for all $x \in A$.*

Lemma 2.3. *Let $C \subseteq X$ be convex and $A \subseteq C$ be such that $K_x(A) = K_x(C)$ for every $x \in A$ (e.g. $C = X$ and A is quasi open). Let $T: A \rightarrow 2^{X^*}$ be a locally efficient, convex, w^* -closed valued monotone operator. For every $x \in A$ let $r_x > 0$ be such that T is r_x -efficient at x and assume that T_{r_x} is w^* -usco at x . Assume also that $T(x) + C_x^* \subseteq T(x)$ for every $x \in A$. Then T is a maximal monotone operator.*

Proof. Assume T is not a maximal monotone operator. Then there exists $(y, y_0^*) \in A \times X^*$ such that $\langle x^* - y_0^*, x - y \rangle \geq 0$ for all $(x, x^*) \in G(T)$, but $y_0^* \notin T(y)$. Since $T(y)$ is convex and w^* -closed, by the separation theorem there exist $u \in X$ and $\alpha \in \mathbb{R}$ such that

$$\langle y^*, u \rangle < \alpha < \langle y_0^*, u \rangle, \text{ for all } y^* \in T(y).$$

First let us prove that $u \in K_y(A) = K_y(C) = \text{cl}(C_y)$. Assume not. Then there exists $z^* \in X^*$ such that

$$\langle z^*, x \rangle \leq 0 < \langle z^*, u \rangle, \text{ for all } x \in C_y.$$

The left inequality implies that $z^* \in C_y^*$ and therefore $nz^* \in C_y^*$ for any positive n . Choose $y^* \in T(y)$. Then $y^* + nz^* \in T(y)$ and therefore $\langle y^* + nz^*, u \rangle < \alpha$ for any positive n , which implies that $\langle z^*, u \rangle \leq 0$, contradicting the choice of z^* . Thus $u \in K_y(A)$.

Next let $r = r_y$ and choose $\beta \in R$ such that $\alpha < \beta < \langle y_0^*, u \rangle$. Let $W = \{z^* \in X^*; \langle z^*, u \rangle < \alpha\}$; since W is w^* -open and contains $T(y)$ and T_r is w^* -usco at y , there exists a neighborhood U of y in X such that $T_r(z) \subseteq W$ for any $z \in U \cap A$. We can also assume that $T_r(z) \neq \emptyset$ for all $z \in U \cap A$ (T is r -efficient at y). Since $u \in K_y(A)$, there exist $t > 0$ and $v \in V = \{z \in X; \langle y_0^*, z \rangle > \beta\}$ such that $\|u - v\| < (\beta - \alpha)/r$ and $y + tv \in A \cap U$. Then $T_r(y + tv) \subseteq W$. For any $u^* \in T_r(y + tv) \subseteq W$ we have

$$0 \leq \langle y_0^* - u^*, y - y - tv \rangle = -t \langle y_0^* - u^*, v \rangle$$

and thus $\langle y_0^*, v \rangle \leq \langle u^*, v \rangle$. This implies that

$$\langle u^*, u \rangle = \langle u^*, v \rangle + \langle u^*, u - v \rangle \geq \langle y_0^*, v \rangle - \|u^*\| \|u - v\| \geq \beta - r(\beta - \alpha)/r = \alpha$$

and thus $u^* \notin W$, contradicting one of our previous choices. The lemma is therefore proved.

Lemma 2.4. *Let $T: A \rightarrow 2^{X^*}$ be a maximal monotone operator and $(x_n) \subset A$ be a sequence norm convergent to $x \in A$. Let also (x_n^*) be a sequence in X^* with $x_n^* \in rB^* \cap T(x_n)$ (for some $r > 0$). Then any w^* -cluster point of (x_n^*) belongs to $T(x) \cap rB^*$ (and there are such cluster points!).*

Proof. Since rB^* is w^* -compact, it follows that (x_n^*) has w^* -cluster points and all are contained in rB^* . Let x^* be such a point. We need to show that $x^* \in T(x)$. Let $y \in A$ and $y^* \in T(y)$. We have

$$\begin{aligned} (*) \quad \langle x^* - y^*, y - x \rangle &= \langle x^* - x_n^*, y - x \rangle + \langle x_n^* - y^*, y - x \rangle \\ &= \langle x^* - x_n^*, y - x \rangle + \langle x_n^* - y^*, y - x_n \rangle + \langle x_n^* - y^*, x_n - x \rangle. \end{aligned}$$

Since x^* is a w^* -cluster point of (x_n^*) , by passing to a subsequence if necessary, we can assume that

$$\lim \langle x^* - x_n^*, y - x \rangle = 0.$$

Since T is monotone,

$$\langle x_n^* - y^*, y - x_n \rangle \leq 0.$$

Finally, since $(x_n^* - y^*)$ is bounded and (x_n) is norm convergent to x , it follows that

$$\lim \langle x_n^* - y^*, x_n - x \rangle = 0.$$

Returning to $(*)$, we get

$$\langle x^* - y^*, y - x \rangle \leq 0$$

and Lemma 2.3 implies that $x^* \in T(x)$.

Corollary 2.5. *If $T: A \rightarrow 2^{X^*}$ is a maximal monotone operator and $r > 0$ is such that $A_r \neq \emptyset$, then T_r is w^* -usco. In particular T is w^* -usco at each point where it is locally bounded.*

Proof. Assume that T_r is not w^* -usco at a point $x \in A$. Then there exist a w^* -neighborhood W of $T_r(x)$, a sequence $(x_n) \subset A$ norm convergent to x

and a sequence (x_n^*) , $x_n^* \in T_r(x_n)$, such that $x_n^* \notin W$ for all n . By Lemma 2.4, there exists a w^* -cluster point $x^* \in T_r(x)$ of (x_n^*) . It follows that there exist n 's such that $x_n^* \in W$. This contradiction proves the lemma.

We are now ready to prove Theorem A and its corollaries.

Proof of Theorem A. Assume that (1) is true. Then, by Lemma 2.2 (2), T is convex and w^* -closed valued. By Corollary 2.5, T_r is w^* -usco for every $r > 0$ for which $A_r \neq \emptyset$. Let $x \in A$ and $x^* \in T(x) + C_x^*$; from Lemma 2.2 (1) (using the maximality of T and the definition of C_x^*), we obtain that $x^* \in T(x)$. Thus (1) implies (2). Obviously (2) implies (3). Finally, Lemma 2.3 shows that (3) implies (1).

Proof of Corollary I. One implication follows from Lemma 2.1 and Corollary 2.5 while the other one follows from the fact that $X_x^* = \{0\}$ for each $x \in X$ and Theorem A (more exactly from (3) $\Rightarrow$ (1)).

Proof of Corollary II. By standard arguments there exists a maximal monotone extension M of T . To prove that M is unique, assume that there exists another maximal monotone operator S on A which contains T . Clearly $M \cap S: A \rightarrow 2^{X^*}$, $(M \cap S)(x) = M(x) \cap S(x)$, is a convex, w^* -closed valued monotone operator on A . It also satisfies $(M \cap S)(x) + C_x^* \subseteq (M \cap S)(x)$ for every $x \in A$ (since we can apply Theorem A to M and S). Now let $r > 0$ be such that $(M \cap S)_r$ is defined. Since $(M \cap S)_r$ has closed graph and is contained in M_r which is w^* -usco (by Theorem A), we can apply Proposition 7.5 in [5] to deduce that $(M \cap S)_r$ is w^* -usco. Finally Theorem A implies that $(M \cap S)$ is a maximal monotone operator on A and therefore $M = M \cap S = S$. This proves the uniqueness of M . The description of M given in the last assertion of the corollary can be proved as in [5], Theorem 7.13.

The proof of Corollary III is obvious and we omit it.

Theorem 2.6. *Let A be a subset of the Banach space X and let $T: A \rightarrow 2^{X^*}$ be a locally efficient maximal monotone operator. Then*

(1) *the function $\psi: A \rightarrow \mathbb{R}$, defined by $\psi(x) = \inf\{\|x^*\|; x^* \in T(x)\}$, is locally bounded and lower semicontinuous on A (here $\|x^*\|$ represents the dual norm);*

(2) *the multivalued map $T_0: A \rightarrow 2^{X^*}$ defined by*

$$T_0(x) = \{x^* \in T(x) : \|x^*\| = \psi(x)\} = T(x) \cap \{x^* \in X^* : \|x^*\| \leq \psi(x)\}$$

is a locally bounded monotone operator and $T_0(x)$ is a nonempty, convex, w^ -compact subset of X^* ;*

(3) *T_0 is w^* -usco at each point where ψ is continuous.*

Proof. (1) Since T is locally efficient, ψ is locally bounded. To prove the lower semicontinuity of ψ , let (x_n) be a sequence in A , norm convergent to x . Let U be a neighborhood of x in A and $r > 0$ be such that $T(y) \cap rB^* \neq \emptyset$ for all $y \in U$. Then there exists $n_0 > 0$ such that $x_n \in U$ if $n > n_0$. For each

$n > n_0$ there exists $x_n^* \in T(x_n)$ such that $\psi(x_n) \geq \|x_n^*\| - 1/n$ and $\|x_n^*\| \leq r$. By Lemma 2.4 the sequence $(x_n^*)_{n > n_0}$ has a w^* -cluster point $x^* \in T(x) \cap rB^*$. Since the norm is w^* -lower semicontinuous,

$$\liminf \psi(x_n) \geq \liminf (\|x_n^*\| - 1/n) \geq \|x^*\| \geq \psi(x),$$

showing that ψ is lower semicontinuous at x .

(2) Let $x \in A$. The definition of ψ implies that for every n there exists $x_n^* \in T(x)$ such that $\psi(x) + 1/n \geq \|x_n^*\|$. This implies that $x_n^* \in (\psi(x) + 1)B^*$ and therefore the sequence (x_n^*) has a w^* -cluster point $x^* \in T(x)$. Since

$$\psi(x) \geq \liminf \|x_n^*\| \geq \|x^*\| \geq \psi(x)$$

$x^* \in T_0(x)$, showing that $T_0(x) \neq \emptyset$. Clearly $T_0(x)$ is convex and w^* -compact. Finally, T_0 is locally bounded because ψ is.

(3) Let ψ be continuous at x and assume that T_0 is not w^* -upper semicontinuous at x . Then there exists a w^* -neighborhood W of $T_0(x)$ such that for every n there exist $x_n \in B(x, 1/n) \cap A$ and $x_n^* \in T_0(x_n)$ but $x_n^* \notin W$. Since T_0 is locally bounded, we may assume that $\|x_n^*\| \leq r$ for n large and some $r > 0$. By Lemma 2.4 there exists a w^* -cluster point x^* of (x_n^*) contained in $T_0(x)$. Since ψ is continuous at x we have

$$\psi(x) = \liminf \psi(x_n) = \liminf \|x_n^*\| \geq \|x^*\|$$

and therefore $x^* \in T_0(x)$. Since W is a w^* -neighborhood of x^* too, there exists n such that $x_n^* \in W$, contradicting the choice of W and x_n^* . Thus T_0 is w^* -upper semicontinuous at x .

Note. The above theorem and its proof have been influenced by [2].

Since a lower semicontinuous function on a Baire space is continuous on a dense G_δ subset of its domain, we obtain

Corollary 2.7. *If A is a Baire subset of X and $T: A \rightarrow 2^{X^*}$ is a locally efficient maximal monotone operator then there exists a dense G_δ subset D of A such that T_0 is w^* -usco at each point of D .*

Proposition 2.8. *Let A be a Baire subspace of X and $T: A \rightarrow 2^{X^*}$ be a maximal monotone operator. Then there exists a dense, relatively open subset D of A such that T is efficient at each point of D .*

Proof. For each positive integer n let

$$F_n = \{x \in A; T(x) \cap nB^* \neq \emptyset\}.$$

Clearly $A = \bigcup F_n$. Observe that each F_n is closed in A . Indeed, let (x_k) be a sequence in F_n norm convergent to $x \in A$. For each k choose $x_k^* \in T(x_k) \cap nB^*$. By Lemma 2.4, the sequence (x_k^*) has a w^* -cluster point $x^* \in T(x) \cap nB^*$, showing that $x \in F_n$.

Now denote by G_n the relative interior of F_n in A . Then, since A is Baire, $D = \bigcup G_n$ is dense in A and of course relatively open. Obviously T is efficient at each point of D .

We are now ready to prove Theorem B and its Corollary.

Proof of Theorem B. From the above proposition, there exists a dense, relatively open subset D_1 of A such that T is efficient at each point of D_1 . By Corollary 2.7 there exists a dense G_δ subset D of D_1 such that, with the notation introduced there, T_0 is w^* -usco at each point of D . Since D is quasi open in X , Corollary I implies that T_0 is a maximal monotone operator on D . However, by construction, $T_0(x) \subseteq T(x)$ for each $x \in A$ and therefore $T_0(x) = T(x)$ for $x \in D$. Finally, using again Corollary I, one can argue as in the proof of Theorem 7.9 in [5] and obtain that $T|D$ is a minimal convex w^* -usco map.

Proof of the corollary. When X is Asplund, this can be proved as in the case of maximal monotone operators with open effective domains (see for example [5]). When X is in Stegall's class $\mathcal{S}$ (see [8]), this follows directly from the definition of the class $\mathcal{S}$.

3. APPLICATIONS TO THE DIFFERENTIABILITY OF CONVEX FUNCTIONS

Let C be a convex subset of the Banach space X such that $\text{claff}(C) = X$, A be a relatively open subset of C and $f: C \rightarrow R$ be convex.

Assume that $f|A$ is locally Lipschitz. Let $a \in A$ and let M be a Lipschitz constant for f on a neighborhood U of a in A . For $x \in U$ construct the continuous, sublinear function $F(x, \cdot): \text{cl}(C_x) \rightarrow R$ as in [9] (recall that $F(x, \cdot)$ is defined on C_x by $F(x, v) = \lim_{t \downarrow 0} (f(x + tv) - f(x))/t$ and then extended by continuity to $\text{cl}(C_x)$). Observe that it is Lipschitz with Lipschitz constant M . The epigraph of $F(x, \cdot)$ and $G = \{(v, r) \in X \times R; -M\|v\| > r\}$ are disjoint convex subsets of $X \times R$ and, since G is open, there exists a closed hyperplane separating them (and passing through $(0,0)$) given by a functional $\Phi \in (X \times R)^*$. Then there exist $\phi \in X^*$ and $a \in R$, $a < 0$, such that $\Phi(v, r) = \phi(v) + ar$. A direct computation shows that $b\phi \in \partial F(x, 0) = \partial f(x)$ and $\|b\phi\| \leq M$, where $b = -1/a$. Thus f is subdifferentiable at every $x \in A$ and the subdifferential map $\partial f: A \rightarrow 2^{X^*}$ is locally efficient. Using arguments similar to those used in the proof of Lemma 2.4 and Corollary 2.5, one can prove that $(\partial f)_r$ is w^* -usco for every $r > 0$. The definition of the subdifferential map also shows that $\partial f(x) + C_x^* \subseteq \partial f(x)$ for each $x \in A$. From Theorem A we obtain that $\partial f: A \rightarrow 2^{X^*}$ is a maximal monotone operator.

Conversely, if $\partial f(x) \neq \emptyset$ for each $x \in A$ and $\partial f: A \rightarrow 2^{X^*}$ is locally efficient, it is easy to see that $f|A$ is locally Lipschitz.

Before stating our last result, let us recall that the convex function $f: C \rightarrow R$ is called *Gâteaux differentiable* at $x \in C$ if, with the above notation, $F(x, \cdot)$ is the restriction of a continuous linear functional f'_x on X ; f is called *Fréchet*

differentiable at $x \in C$ if there exists $f'_x \in X^*$ such that

$$\lim(f(y) - f(x) - \langle f'_x, y - x \rangle) / \|y - x\| = 0,$$

as y tends to x in $C \setminus x$.

Proposition 3.1. *Let C be a convex subset of the Banach space X such that $\text{claff}(C) = X$, A be a relatively open subset of C and $f: C \rightarrow \mathbb{R}$ be convex.*

(1) *$f|A$ is locally Lipschitz if and only if $\partial f(x) \neq \emptyset$ for every $x \in A$ and $\partial f: A \rightarrow 2^{X^*}$ is locally efficient; if this is the case, then ∂f is a maximal monotone operator on A ;*

(2) *if A is a Baire space and $f|A$ is locally Lipschitz then there exists a dense G_δ subset D of A such that $\partial f(x)$ lies in a face of a sphere of X^* for every $x \in D \cap N(C)$.*

(3) *if X is Asplund (resp. of class $\mathcal{S}$), A is Baire and $f|A$ is locally Lipschitz then there exists a dense G_δ subset A_0 of A such that f is Fréchet (resp. Gâteaux) differentiable at each point of A_0 .*

Proof. We already proved (1). The second assertion follows from Theorem B and the fact that $N(C)$ is a quasi open subset of X . To prove (3), notice that A is paracompact (being a metric space). Using a partition of the unity argument and the fact that ∂f is locally efficient one can construct a continuous function $\varphi: A \rightarrow \mathbb{R}$ such that $\partial_\varphi f: A \rightarrow 2^{X^*}$ defined by $(\partial_\varphi f)(x) = \partial f(x) \cap \varphi(x)B^*$ is nonempty valued. As in the proof of Theorem 2.6 one can check that $\partial_\varphi f$ is a convex w^* -usco map. Let $T: A \rightarrow 2^{X^*}$ be a minimal convex w^* -usco map whose graph is contained in the graph of $\partial_\varphi f$. Assume now that X is Asplund. Then (see for example [5, Lemma 7.14 and Theorem 5.7]) there exists a selection $\sigma: A \rightarrow X^*$ for T which is norm-to-norm continuous at each point of a dense G_δ subset A_0 of A . Clearly σ is a selection for ∂f too. It remains to notice that the “if” part of Proposition 2.8 in [5] is also true in our context, namely: the existence of a norm-to-norm continuous at x selection for the subdifferential map of f implies the Fréchet differentiability of f at x . The case when X is of class $\mathcal{S}$ can be treated similarly.

Note. When C is open, the above results are well known. For a not necessarily open C , parts of this proposition were previously proved by different methods in [9, 6, 4 and 10].

Note. After this paper was accepted for publication, we noticed the following direct consequence of Theorem A:

Theorem A'. *Let C , A and T be as in Theorem A and let $S: A \rightarrow 2^{X^*}$ be any locally bounded, convex valued, w^* -usco map such that $S(x) \subseteq T(x)$ for all $x \in A$ (such an S always exists if T is maximal monotone). Then T is maximal monotone if and only if $T(x) = S(x) + C_x^*$ for all $x \in A$.*

This last result is closely related to some results due to J. Borwein, S. Fitzpatrick and P. Kenderov (Minimal convex usc's and monotone operators on small sets, preprint, 1989).

ACKNOWLEDGMENT

We would like to thank Professor R. R. Phelps for the useful remarks he made after reading a first version of this paper.

REFERENCES

1. J. M. Borwein and S. P. Fitzpatrick, *Local boundedness of monotone operators under minimal hypotheses*, Bull. Austral. Math. Soc. (to appear).
2. P. S. Kenderov, *The set-valued monotone mappings are almost everywhere single-valued*, C. R. Acad. Bulgare Sci. **27** (1974), 1173–1175.
3. ———, *Monotone operators in Asplund spaces*, C. R. Acad. Bulgare Sci. **30** (1977), 963–964.
4. D. Noll, *Generic Fréchet differentiability of convex functions on small sets*, preprint (1987).
5. R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Math., vol. 1364, Springer-Verlag, 1989.
6. J. Rainwater, *Yet more on the differentiability of convex functions*, Proc. Amer. Math. Soc. **103** (1988), 773–778.
7. R. T. Rockafellar, *Local boundedness of nonlinear monotone operators*, Mich. Math. J. **16** (1969), 397–407.
8. C. Stegall, *A class of topological spaces and differentiation of functions on Banach spaces*, Proc. Conf. on Vector Measures and Integral Representations of Operators, Vorlesungen aus dem Fachbereich Math., Heft **10** (W. Ruess, ed.), Essen, 1983.
9. M. E. Verona, *More on the differentiability of convex functions*, Proc. Amer. Math. Soc. **103** (1988), 137–140.
10. ———, *On the differentiability of convex functions*, Proceedings of the Centre of Mathematical Analysis, vol. 20, Canberra, 1988, 195–202.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CALIFORNIA STATE UNIVERSITY, LOS ANGELES, CALIFORNIA 90032

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125

Current address (Maria Elena Verona): DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92521