ROTATION SETS OF TORAL FLOWS

JOHN FRANKS AND MICHAŁ MISIUREWICZ

(Communicated by Kenneth R. Meyer)

ABSTRACT. We consider the rotation set $\rho(\Phi)$ for a lift $\Phi = \{\Phi_t\}_{t \in \mathbb{R}}$ of a flow $\varphi = \{\varphi_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2\}_{t \in \mathbb{R}}$. Our main result is that $\rho(\Phi)$ consists of either a single point, a segment of a line through 0 with rational slope, or a line segment with irrational slope and one endpoint equal to 0. Any set of one of these types is the rotation set for some flow.

In this article we consider the rotation set as defined in [MZ], for a lift $\Phi = \{\Phi_t\}_{t \in \mathbb{R}}$ of a flow $\varphi = \{\varphi_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2\}_{t \in \mathbb{R}}$.

(1.1) **Definition.** The rotation set $\rho(\Phi)$ of a flow Φ on the plane is defined by $v \in \rho(\Phi)$ if and only if there are sequences $x_i \in \mathbb{R}^2$ and $t_i \in \mathbb{R}^+$ with $\lim_{i \rightarrow \infty} t_i = \infty$ such that

$$\lim_{i \rightarrow \infty} \frac{\Phi_{t_i}(x_i) - x_i}{t_i} = v.$$

It is easy to see that the rotation set is a compact set, and in [MZ] it is proven that it is convex.

Our main result is the following.

(1.2) **Theorem.** If φ is a continuous flow on $\mathbb{T}^2$ and Φ is its lift to $\mathbb{R}^2$, then the rotation set $\rho(\Phi)$ is one of the following:

- (a) A set consisting of a single point $v \in \mathbb{R}^2$,
- (b) A segment of a line passing through 0 and some other point of $\mathbb{Q}^2$. (The segment need not contain 0.)
- (c) A line segment with one end at $0 \in \mathbb{R}^2$ and having irrational slope.

It is easy to obtain examples of type (a) and (b). For example, any constant vector field on $\mathbb{T}^2$ will have a rotation set consisting of a single point, and an example of type (b) can be constructed by starting with a constant vector field with rational slope and multiplying it by a function which is constant on each orbit but with some different values on different orbits.

An example of type (c) is somewhat more subtle. We only sketch the idea. More details can be found in a paper of Handel [H2] who attributes the example

Received by the editors May 20, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 58F22; Secondary 58F25.

©1990 American Mathematical Society
 0002-9939/90 \$1.00 + \$.25 per page

to Katok. Start with a constant vector field on $\mathbb{T}^2$ with irrational slope and multiply by a function which vanishes at only one point, say 0. Let C be a circle in $\mathbb{T}^2$ with 0 slope and containing 0. Then for every point $x \in C$ except 0 there is a well-defined return time $\sigma(x) > 0$. If the original scaling function is properly chosen, the function $\sigma : C \rightarrow \mathbb{R}^+$ has a finite integral. Using the ergodic theorem it is then easy to see there will be points of C with nonzero rotation vector parallel to the original vector field. Clearly 0 is also a rotation vector. Thus from convexity of the rotation vector it follows that this example is of type (c).

Proof of (1.2). Let S denote the set of $x \in \mathbb{R}^2$ such that the limits

$$\lim_{t \rightarrow +\infty} \frac{\Phi_t(x) - x}{t}$$

and

$$\lim_{t \rightarrow -\infty} \frac{\Phi_t(x) - x}{t}$$

exist, are equal and nonzero. For $x \in S$ denote

$$v(x) = \lim_{t \rightarrow \infty} \frac{\Phi_t(x) - x}{t}.$$

By continuity it is easy to see that $\rho(\Phi)$ is equal to the rotation set of the time one map $\rho(\Phi_1)$. Hence by a result of [MZ] for every extreme point v of $\rho(\Phi)$, there exists an ergodic probability measure ν , invariant under φ_1 and such that Φ_1 has mean rotation vector v with respect to ν .

The measure

$$\mu = \int_0^1 (\varphi_t)_*(\nu) dt$$

(where $(\varphi_t)_*$ denotes the operator induced on the space of measures) is an ergodic probability measure invariant under φ and the mean rotation vector for φ with respect to μ is v . By the ergodic theorem, if $v \neq 0$, then for almost all points x with respect to μ , every point $y \in \pi^{-1}(x)$ belongs to S and $v(y) = v$. Therefore, since any extreme points of $\rho(\Phi)$ other than 0 will be realized in S , in order to prove the theorem it is enough to study Φ restricted to S .

Suppose $x \in S$ and $v(x)$ has a nonzero second coordinate. Then the trajectory of x intersects each horizontal line and does so only for a bounded set of times (i.e., if the line is L then the set $\{t | \Phi_t(x) \in L\}$ is bounded). Therefore we can define an "up" orientation on the trajectory of x . This is the orientation given by the flow if the second coordinate of $v(x)$ is positive and the reverse orientation otherwise. With respect to this orientation, for every horizontal line L we can find the first and the last points of intersection of the trajectory with L . Denote these points by $a(x, L)$ and $b(x, L)$ respectively. If two points x, y lie on the same horizontal line, we shall write $x < y$ if x lies to the left of y . Similarly we shall use the notations $x > y$, $x \leq y$, and $x \geq y$.

(1.3) **Lemma.** Suppose L, M are horizontal lines, $x, y \in S$, and both $v(x)$ and $v(y)$ have nonzero second coordinates. Then the following four conditions are equivalent:

- (a) $a(x, L) < a(y, L)$,
- (b) $a(x, M) < a(y, M)$,
- (c) $b(x, L) < b(y, L)$,
- (d) $b(x, M) < b(y, M)$.

Proof. Let t be so large that $\Phi_s(x) \notin L \cup M$ and $\Phi_s(y) \notin L \cup M$ for any s with $|s| \geq t$. Take horizontal lines K and N so low below L and M and so high above L and M respectively that $\Phi_s(x) \notin K \cup N$ and $\Phi_s(y) \notin K \cup N$ for any s with $|s| \leq t$.

If we go along the trajectory of x in the “upward” direction, we pass through $b(x, L)$ before we pass through $a(x, N)$. Denote the part of the trajectory of x between these two points by α . Then $\alpha \cap L = \{b(x, L)\}$ and $\alpha \cap N = \{a(x, N)\}$. Denote by L^+ and N^+ the semilines $\{z \in L | z \geq b(x, L)\}$ and $\{z \in N | z \geq a(x, N)\}$ respectively. If $b(x, L) < b(y, L)$ then at $b(y, L)$ the trajectory of y enters (when we move “upward”) the region A bounded by L^+ , α , and N^+ , and it can leave it only through a point of N^+ . It cannot intersect N earlier, so we get $a(x, N) < a(y, N)$. Analogously we see that if $b(y, L) < b(x, L)$ then $a(y, N) < a(x, N)$. This proves that (c) is equivalent to $a(x, N) < a(y, N)$. The same argument shows that (d) is equivalent to $a(x, N) < a(y, N)$ so (c) is equivalent to (d). In a similar manner one shows that (a) and (b) are both equivalent to $b(x, K) < b(y, K)$. Finally the same argument again shows that $a(x, N) < a(y, N)$ is equivalent to $b(x, K) < b(y, K)$ so (a), (b), (c), and (d) are all equivalent.

The following proposition follows from results of M. Handel [H] in the case that the flow has at least one rest point. Handel’s results are more general in the sense that they apply to a map of entropy zero, not just the time one map of a flow, but they are correspondingly more difficult and require a surface of negative genus (which can be obtained by puncturing the torus at rest points of the flow).

(1.4) **Proposition.** If S is not empty then there exists a line M in $\mathbb{R}^2$ with the property that each trajectory of Φ has all of its points within a bounded distance of M . The bound depends on the trajectory.

Proof. By changing coordinates on the torus if necessary, (for example by a linear shear) we can assume that there is $x \in S$ with $v(x)$ having nonzero slope $1/\gamma$. Let L be a horizontal line and define $L_n = L + (0, n)$ for all $n \in \mathbb{Z}$. If $z = a(x, L)$ we will show first that for all $m \in \mathbb{Z}$,

$$(1) \quad \|a(z, L_m) - u(m)\| \leq 3,$$

where $u(t)$ is the line $u(t) = z + (\gamma t, t)$.

If $m = 0$, then (1) is obvious. Assume that $m > 0$. Let $A_n = \{a(z', L_n) | z' = z + (p, q), (p, q) \in \mathbb{Z}^2\}$, and let $T_m : \bigcup_{n \in \mathbb{Z}} A_n \rightarrow \bigcup_{n \in \mathbb{Z}} A_n$ be defined by

$T_m(y) = a(y, L_{n+m})$ when $y \in A_n$. Then T_m restricted to A_n is an order preserving bijection onto A_{m+n} as a consequence of Lemma (1.3). It follows easily from the definition that $T_m \circ T_p = T_{m+p}$ and that for any $(p, q) \in \mathbb{Z}^2$ we have

$$(2) \quad T_m(y + (p, q)) = T_m(y) + (p, q).$$

For a given m there is $k \in \mathbb{Z}$ such that $k \leq (T_m(z) - z)_1 \leq k + 1$ (where $(\cdot)_1$ denotes the first coordinate). Since $T_m(z + (1, 0)) = T_m(z) + (1, 0)$ it follows that whenever $z \leq y \leq z + (1, 0)$ then

$$(3) \quad k - 1 \leq (T_m(y) - y)_1 \leq k + 2.$$

As a consequence of (2) it follows that (3) holds for all $y \in \bigcup_{n \in \mathbb{Z}} A_n$.

From this it follows that for all $p > 0$,

$$(T_{mp}(z) - z)_1 = (T_m^p(z) - z)_1 = \sum_{i=0}^{p-1} (T_m(T_m^i(z)) - T_m^i(z))_1 \leq p(k + 2).$$

Therefore $\gamma \leq (k + 2)/m$. Similarly we can show that $\gamma \geq (k - 1)/m$ and hence that $k - 1 \leq \gamma m \leq k + 2$. This, together with (3) establishes (1). The case $m < 0$ is analogous.

Let α denote the arc of the trajectory of Φ through z and between $a(z, L)$ and $b(z, L_1)$, and let c denote the diameter of α . It is then easy to see that if x_1, x_2 are on a trajectory of Φ and between L_n and L_{n+1} , then $\|x_1 - x_2\| < c + 2$. Otherwise these points could be moved by an integer lattice translate so that they are between L and L_1 and so they straddle the arc α , i.e. so that they are in different components of the complement of the trajectory of Φ through z . Since x_1 and x_2 are on the same trajectory this would clearly be impossible.

It now follows that if y is any point of the trajectory of Φ through z , then for some m we have y between L_m and L_{m+1} so

$$\|y - u(m)\| \leq \|y - a(z, L_m)\| + \|a(z, L_m) - u(m)\| \leq c + 5,$$

by (1). Thus we have established the lemma for the trajectory through z .

Now if y is an arbitrary point in $\mathbb{R}^2$, then there are $k_1, k_2, m \in \mathbb{Z}$ such that the points $y - (k_1, m)$ and $y - (k_2, m)$ are between L and L_1 and they straddle α . Thus the trajectory through y is entirely contained between the trajectories through $z + (k_1, m)$ and $z + (k_2, m)$. Since these trajectories are translates of the trajectory through z by the vectors (k_1, m) and (k_2, m) respectively, it follows that the trajectory through y must be a bounded distance from the line $u(t)$.

The following corollary follows immediately from Proposition (1.4). It is quite similar to a result of [M].

(1.5) **Corollary.** *If $x, y \in S$ then $v(x)$ and $v(y)$ are linearly dependent.*

As we remarked above a nonzero extreme point of $\rho(\Phi)$ is always $v(x)$ for some $x \in S$. Thus by Corollary (1.5), all extreme points of $\rho(\Phi)$ and hence the whole rotation set $\rho(\Phi)$ are contained in some line passing through 0. Therefore we have three possible cases:

Zero case. $\rho(\Phi) = \{0\}$.

Rational case. $\rho(\Phi) \neq \{0\}$ and $\rho(\Phi)$ is contained in a line through 0 with rational slope.

Irrational case. $\rho(\Phi) \neq \{0\}$ and $\rho(\Phi)$ is contained in a line through 0 with irrational slope.

In the zero case we have (a) of the Theorem. In the rational case we have either (a) or (b) of the Theorem. Therefore it remains to investigate the irrational case.

From now on we shall assume that we are in the irrational case. In particular, we assume S is nonempty and the ratio γ of the first and the second coordinates of each $v(x)$, $x \in S$, is independent of x and irrational.

(1.6) **Proposition.** *If $v(x)$ has irrational slope for some $x \in S$ then there exists at most one φ -invariant ergodic probability measure which has a non-zero rotation vector.*

Proof. Let L be a horizontal line in $\mathbb{R}^2$ and let $A_L = \{a(y, L) | y \in S\}$. The map $T_L : A_L \rightarrow A_L$ defined by

$$T_L(a(x, L)) = a(x, L + (0, 1)) - (0, 1)$$

maps A_L onto itself, and has the property that

$$T_L(x + (k, 0)) = T_L(x) + (k, 0)$$

for all $k \in \mathbb{Z}$. It preserves the ordering on A_L by Lemma (1.3) and for every $x \in A_L$ we have (by Corollary (1.5) and the definition of γ)

$$\lim_{n \rightarrow \infty} \frac{T_L^n(x) - x}{n} = \gamma.$$

Therefore if $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is the natural projection, and $X = \pi(A_L) \subset \pi(L)$, we have a map $T : X \rightarrow X$ such that $T \circ \pi = \pi \circ T_L$.

Every point $x \in S$ can be uniquely represented as $\Phi_t(y)$ where $y \in A_L + (0, k)$ for some $k \in \mathbb{Z}$ and $0 \leq t < \tau(y)$ where $\tau(y)$ is such that

$$\Phi_{\tau(y)}(y) = a(y, L + (0, k + 1)).$$

The dependence of y and $\tau(y)$ on x is Borel. If $\pi(y) = \pi(y')$ then clearly $\tau(y) = \tau(y')$, so we have $\tau = \sigma \circ \pi$ for a Borel function $\sigma : X \rightarrow \mathbb{R}^+$, which is the "return time" for points of X . The function σ is probably not continuous, but by the continuity of φ it is bounded away from 0, so $1/\sigma$ is integrable for every finite measure on X .

Passing to the torus $\mathbb{T}^2$, we get a map $r : S \rightarrow Y = \{(z, t) | z \in X, 0 \leq t < \sigma(z)\}$, which is a special representation of the flow φ for any φ -invariant probability measure μ on S (as the measure on Y we take $T_*(\mu)$), see [CFS, Chapter 11]. This gives us a finite measure ν on X , invariant under T . On the other hand, the measure ν and the function σ determine the measure μ . Therefore if there are two distinct probability measures μ_1, μ_2 , which are φ -invariant, then the corresponding measures ν_1 and ν_2 on X are distinct. Since $\int \sigma d\nu_i = \mu_i(S) = 1$, the measures $\nu_i/\nu_i(X)$ are distinct.

Hence, we need only show that there are no two distinct T -invariant probability measures on X . The set X is a subset of the circle $\pi(L)$ and T_L is a lift which is strictly monotone and has an irrational rotation vector. If we try to extend T to a homeomorphism of the whole circle, we can meet only two types of obstructions. The first one is caused by discontinuities, i.e., by points which have whole intervals as images. The second one is caused by whole intervals being mapped to points. We can remove these obstructions by blowing up respectively backward and forward semiorbits of the points which cause troubles. Then we obtain a homeomorphism of the circle and nothing will change with the invariant measures, since we have blown up at most countably many points and the measures are non-atomic due to the irrational rotation number. However, it is known that an orientation preserving homeomorphism of the circle with an irrational rotation number is uniquely ergodic (see [CFS, Chapter 3, §3, Theorem 5]). This completes the proof of Proposition (1.6).

Now from Proposition (1.6) it follows that in the irrational case we have either (a) or (c) of the Theorem. This completes the proof of Theorem (1.2).

By the *pointwise rotation set* of Φ we shall mean the set

$$\rho_p(\Phi) = \bigcup_{x \in \mathbb{R}^2} \rho(\Phi, x),$$

where $\rho(\Phi, x)$ is the set of all accumulation points of $\Phi_t(x)$ as $t \rightarrow \infty$.

Problem. *What can one say about the pointwise rotation set $\rho_p(\Phi)$ when the set $\rho(\Phi)$ of a segment of a line with irrational slope?*

Perhaps this problem can be answered by studying a constant vectorfield on $\mathbb{T}^2$ multiplied by a function ψ with one zero and $\int 1/\psi$ finite. The problem for such flows can be reduced to the following:

Suppose r is an irrational rotation of S^1 and $\sigma : S^1 \rightarrow \mathbb{R}^+ \cup \{\infty\}$ is an integrable function such that $\sigma(x) = \infty$ at exactly one point and $1/\sigma$ is continuous. What can one say about the accumulation points of the sequence of ergodic averages

$$\frac{1}{n} \sum_{k=0}^{n-1} \sigma(r^k(z))?$$

By the ergodic theorem the limit exists and equals $\int \sigma$ for almost all z . On the other hand it is ∞ for a dense set (preimages of x , where $\sigma(x) = \infty$).

It is not difficult to prove that on a dense G_δ set, the lim sup of the ergodic averages is ∞ and the lim inf is $\int \sigma$.

Conjecture. *For any homeomorphism of the two-dimensional torus having a rotation set which is contained in a line, that rotation set is one of the three types of Theorem (1.2), translated by vector from $\mathbb{Q}^2$.*

REFERENCES

- [CFS] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic theory*, Springer-Verlag, New York, Berlin, Heidelberg, 1982.
- [H] M. Handel, *Zero entropy surface diffeomorphisms*, preprint.
- [H2] ———, *Periodic point free homeomorphisms of T^2* , preprint.
- [M] N. G. Markley, *The Poincaré–Bendixson Theorem for the Klein bottle*, Trans. Amer. Math. Soc. **135** (1969), 159–165.
- [MZ] M. Misiurewicz and K. Ziemian, *Rotation sets of maps of tori*, J. London Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60208-2730

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, PKiN IX p., 00-901 WARSZAWA, POLAND