AN UNBOUNDEDNESS PROPERTY FOR NORMS OF LENGTH $\geq \omega_2$

STEVE JACKSON

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ABSTRACT. We prove from the axiom of determinacy that for every norm φ from a set of reals A onto ω_2 there is a Σ_1^1 subset of A coding uncountably many ordinals. This extends a result of Kechris.

We prove a result which extends a theorem of Kechris (see [2]), but first recall some basic notions. The reader wishing more details can consult ([3]). We let $\mathbf{R} = \omega^\omega$ be the set of “reals”, i.e. the set of infinite sequences of natural numbers. To each $A \subseteq \mathbf{R}$, we associate a two player game where the players alternately play natural numbers, thereby jointly producing a real $x \in \omega^\omega$, and player I wins if $x \in A$. The axiom of determinacy, AD, asserts that for every such A , one of the players has a winning strategy, where the notion of winning strategy is defined in the usual way. DC is the axiom of dependent choice which asserts that every ill-founded binary relation has an infinite descending chain. Throughout this paper, we will work in the theory $\text{AD} + \text{DC}$. By a *norm* φ on a set $A \subseteq \mathbf{R}$, we simply mean a map $\varphi: A \xrightarrow{\text{onto}} \alpha$, for some $\alpha \in \text{ON}$. If α is regular, we may identify any unbounded map $\varphi: A \rightarrow \alpha$ with an onto map. We say $B \subseteq A$ is *bounded* in the norm if for some $x \in A$ and all $y \in B$, $\varphi(y) < \varphi(x)$.

Modifying slightly a definition of Kechris, we say a norm $\varphi: A \rightarrow \kappa$ has the *strong boundedness property* if for every $B \subseteq A$, $B \in \Sigma_1^1$, $\{\varphi(x): x \in B\}$ is countable. (Kechris used “compact” in the definition instead of Σ_1^1 —see the comments at the end of this paper). As Kechris points out, clearly for any $\kappa < \omega_2$ there are such norms. Kechris proves that for any such norm we must have $\kappa < \delta_3^1 (= \omega_{\omega+1}$ assuming AD). It was left open for which $\kappa < \delta_3^1$ such norms exists. We extend his result by showing that no such norms exist for any $\kappa \geq \omega_2$ (it is, of course, enough to rule out $\kappa = \omega_2$ here). We may also state

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our result in a “positive” manner as follows:

Theorem. *For any $A \subseteq \omega^\omega$ and norm $\varphi: A \xrightarrow{\text{onto}} \omega_2$, there is a $\Sigma_1^1 B \subseteq A$ such that $\{\varphi(x): x \in B\}$ is uncountable.*

We first show the following:

Theorem 1 (AD + DC). *There is a Π_2^1 -complete set A and a norm $\varphi: A \rightarrow \omega_2$ satisfying:*

(1) *Each $B \subseteq A$ which is Σ_2^1 is bounded in the norm.*

(2) *Each $B \subseteq A$ which is Σ_2^1 can be covered in the norm by a closed set $B' \subseteq A$. That is, $\forall x \in B \exists y \in B' \varphi(x) = \varphi(y)$.*

Proof. Let $\text{WO} \subseteq \omega^\omega$ be the standard set of codes for countable ordinals, a complete Π_1^1 -set. Let $A = \{\sigma: \exists n(\forall w(w \in \text{WO} \Rightarrow \sigma_n(w) \in \text{WO}))\}$, where σ_n is the n th component of σ with respect to some standard coding of $\mathbf{R}^\omega$ into $\mathbf{R}$, and $\sigma_n(w)$ is the result of applying σ_n to w , where we view reals as giving strategies for player II in an integer game. To be specific, let $\tau_0, \tau_1, \dots, \tau_m, \dots$ enumerate $\omega^{<\omega}$ such that any sequence τ precedes its extensions in the enumeration. Then for any $x \in \omega^\omega$, we set $x(\tau_n) = x(n)$, where $x = (x(0), x(1), \dots)$. Clearly $A \in \Pi_2^1$.

Claim. A is Π_2^1 -complete.

Proof. It is enough to show that $A' = \{\sigma: \forall w(w \in \text{WO} \Rightarrow \sigma(w) \in \text{WO})\}$ is Π_2^1 -complete, as any reduction of a Π_2^1 -set to A' easily yields a reduction to A . If A' were not Π_2^1 -complete, then by Wadge, A' would be Σ_2^1 . We fix some notation: We let $T \subseteq (\omega \times \omega_1 \times \omega \times \omega \times \omega)^{<\omega}$ be the canonical “Kunen Tree” defined as follows: (as usual, we identify elements of T with elements of $\omega^{<\omega} \times \omega_1^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ with each component having the same length). We let $\bar{T} \subseteq (\omega \times \omega_1)^{<\omega}$ be a tree witnessing WO is ω_1 -Suslin, and $S \subseteq (\omega \times \omega)^{<\omega}$ a tree witnessing the complement of WO , a complete Σ_1^1 set, is ω -Suslin. Then a sequence $(\vec{x}, \vec{\alpha}, \vec{\sigma}, \vec{y}, \vec{z}) \in T$ iff $\text{lh}(\vec{x}) = \text{lh}(\vec{\alpha}) = \dots = \text{lh}(\vec{z}) = n$ for some n and $\exists x \in \omega^\omega \exists \alpha \in \omega_1^\omega \exists \sigma \in \omega^\omega \exists y \in \omega^\omega \exists z \in \omega^\omega$ (x extends $\vec{x} \wedge \alpha$ extends $\vec{\alpha} \wedge z$ extends $\vec{z} \wedge (x, \alpha) \in [\bar{T}] \wedge \sigma(x) = y \wedge (y, z) \in [S]$). Here $[\bar{T}]$ denotes the set of paths through $\bar{T}$, and similarly for S . For $\sigma \in \omega^\omega$ we let T_σ denote that section of T at σ , $T_\sigma \subseteq (\omega \times \omega_1 \times \omega \times \omega)^{<\omega}$. That is, $(\vec{x}, \vec{\alpha}, \vec{y}, \vec{z}) \in T_\sigma$ iff $\text{lh}(\vec{x}) = \text{lh}(\vec{\alpha}) = \dots = \text{lh}(\vec{z}) = n$ for some n and $(\vec{x}, \vec{\alpha}, \sigma \upharpoonright n, \vec{y}, \vec{z}) \in T$. By standard arguments, due to Kunen, for all $f: \omega_1 \rightarrow \omega_1 \exists \sigma \in \omega^\omega$ such that T_σ is well-founded and for almost all $\alpha < \omega_1$ with respect to the ω -cofinal normal measure on ω_1 , $f(\alpha) < |T_\sigma \upharpoonright \alpha| =$ the rank of T_σ restricted to α (play the game where I plays x , II plays y , and II wins iff $x \in \text{WO} \Rightarrow S_y$ is well-founded of rank $> f(|x|)$, where $|x| < \omega_1$ is the ordinal coded by $x \in \text{WO}$. II wins by boundedness; a winning strategy σ for II will work). It follows, in particular that $\{|T_\sigma|: T_\sigma \text{ is well-founded}\}$ is unbounded in ω_2 , as $|T_\sigma|$ is represented in

the ultrapower by the normal measure on ω_1 by the function $\alpha \rightarrow |T_\sigma \restriction \alpha|$ by normality, and $\omega_2 \leq$ (in fact $=$) the ultrapower of ω_1 by the normal measure. If A' were Σ_2^1 , then A' would be ω_1 -Suslin, say witnessed by the tree $\mathcal{U}$ on $\omega \times \omega_1$. However, $A' = \{\sigma : \forall w (w \in \text{WO} \Rightarrow \sigma(w) \in \text{WO})\} = \{\sigma : T_\sigma \text{ is well-founded}\}$. We then consider the tree V on $\omega \times \omega_1 \times \omega \times \omega_1 \times \omega \times \omega$ from “putting together” the trees $\mathcal{U}$ and the T_σ defined as follows: $(\vec{\sigma}, \vec{\beta}, \vec{x}, \vec{\alpha}, \vec{y}, \vec{z}) \in V$ iff $\text{lh}(\vec{\sigma}) = \dots = \text{ln}(\vec{z}) \wedge \exists \sigma$ extending $\vec{\sigma} \exists \beta$ extending $\vec{\beta} \dots \exists z$ extending $\vec{z} ((\sigma, \beta) \in [\mathcal{U}] \wedge (x, \alpha, y, z) \in [T_\sigma])$. It follows that V is well founded, yet $|V| > |T_\sigma|$ for all $\sigma \in A'$, a contradiction. Hence, A is Π_2^1 -complete.

We define the norm φ on A now as follows: for $\sigma \in A$ we set $\varphi(\sigma) = \inf_{n : \sigma_n \in A'} (|T_{\sigma_n}|) < \omega_2$. Now, φ as defined may not map onto ω_2 , but as ω_2 is regular, we may identify σ with an onto map.

We isolate the combinatorial construction used in our proof in the following lemma:

Lemma. *If $B \subseteq A$ is Σ_2^1 , then $\sup_{\sigma \in B} (\varphi(\sigma)) < \omega_2$.*

Proof. We define a tree $\mathcal{T}$ on $\omega \times \omega_1 \times \omega \times \omega \times \omega$ such that, roughly, $\mathcal{T}_\sigma$, will be the “diagonal direct product” of the trees $T_{\sigma_1}, T_{\sigma_2}, \dots, T_{\sigma_n}, \dots$. Via some fixed coding (which we suppress here), we may think of elements $S = (\vec{x}, \vec{\alpha}, \vec{y}, \vec{z}) \in (\omega \times \omega_1 \times \omega \times \omega)^m$ as coding m many elements $S_1 = (\vec{x}_1, \vec{\alpha}_1, \vec{y}_1, \vec{z}_1) \in (\omega \times \omega_1 \times \omega \times \omega)^m$, $S_2 = (\vec{x}_2, \vec{\alpha}_2, \vec{y}_2, \vec{z}_2) \in (\omega \times \omega_1 \times \omega \times \omega)^{m-1}, \dots, S_m = (\vec{x}_m, \vec{\alpha}_m, \vec{y}_m, \vec{z}_m) \in (\omega \times \omega_1 \times \omega \times \omega)$. We do this in such a manner so that S' extends S (where $\text{length } S = m$) iff S'_1 extends S_1 , S'_2 extends $S_2, \dots, S'_m$ extends S_m . If $\tau = (\vec{x}, \vec{\alpha}, \vec{\sigma}, \vec{y}, \vec{z}) \in (\omega \times \omega_1 \times \omega \times \omega \times \omega)^m$, we define τ to be in $\mathcal{T}$ iff $S \equiv (\vec{x}, \vec{\alpha}, \vec{y}, \vec{z})$ codes sequences $S_1, \dots, S_m$ as above and $\exists \sigma$ extending $\vec{\sigma}$ such that for all $1 \leq i \leq m$, $(\sigma \restriction m - \widehat{i} + 1, S_i) \in T$ where the hat denotes the permutation map defined by $(\vec{\sigma}, \vec{x}, \widehat{\vec{\alpha}}, \vec{y}, \vec{z}) = (\vec{x}, \vec{\alpha}, \vec{\sigma}, \vec{y}, \vec{z})$. It is easy to see that $\sigma \in A$ iff $\mathcal{T}_\sigma$ is well founded, using essentially the fact that a direct product of trees is well founded iff one of its factors is. Let now $B \subseteq A$ be Σ_2^1 , and $\mathcal{U}$ be a tree on $\omega \times \omega_1$ witnessing that B is ω_1 -Suslin. We form the tree V on $\omega \times \omega_1 \times \omega \times \omega_1 \times \omega \times \omega$ by “putting together” the trees for $\mathcal{U}$ and the tree $\mathcal{T}_\sigma$, as we have done earlier. Thus, V is well founded, and so has rank $< \omega_2$. However, we claim that for each $\sigma \in A$, $|\mathcal{T}_\sigma| \leq \varphi(\sigma) + \omega \leq |\mathcal{T}_\sigma| + \omega \cdot 2$. This is easy to see from the definition of $\mathcal{T}$, using essentially the fact that the rank of a product of trees is the infimum of their ranks. We need the extra ω 's since we are using the diagonal direct product instead of the product. However, it is clear that for each σ that $|V_\sigma| \geq |\mathcal{T}_\sigma|$. Hence, $\sup_{\sigma \in B} |\mathcal{T}_\sigma| < \omega_2$ which implies that $\sup_{\sigma \in B} \varphi(\sigma) < \omega_2$. This establishes the lemma.

Using this A and norm φ , we have now verified the first part of our theorem. We now show that any $B \subseteq A$, $B \in \Sigma_2^1$, can be covered in norm by a $B' \subseteq A$,

$B' \in \Sigma_1^1$. (Note: by the coding lemma (see [3]) every bounded set $S \subseteq \omega_2$ can be covered in the norm by a Σ_2^1 set, C , i.e. $S = \{\varphi(\sigma) : \sigma \in C\}$. So, fix $B \subseteq A$, $B \in \Sigma_2^1$.

Since $\neg B$ is Π_2^1 and A' is Π_2^1 -complete, we let $\pi \in \omega^\omega$ continuously reduce $\neg B$ to A' , i.e. for all $z \in \omega^\omega$, $z \in \neg B \Leftrightarrow \pi(z) \in A'$. For each $x \in \omega^\omega$ we now define a real $\hat{x}$ by: $(\hat{x})_1 = \pi(x)$, $(\hat{x})_2 = (x)_1$, $(\hat{x})_3 = (x)_2$, $\dots$, etc. The function defined by $f(x) = \hat{x}$ is easily seen to be continuous.

Claim. For all $x \in \omega^\omega$, $\hat{x} \in A$.

Proof. If $x \in B \subseteq A$, then some $(x)_n \in A'$, so $(\hat{x})_{n+1} \in A'$ and we are done. If $x \notin B$, then $\pi(x) \in A'$, so $(\hat{x})_1 \in A'$ and we are done.

Hence, if we let $B' = \{\hat{x} : x \in \omega^\omega\}$ then $B' \subseteq A$, and B' is Σ_1^1 . It remains to show that B' covers B . However, if $x \in B$ then $(\hat{x})_1 = \pi(x) \notin A'$, and so $\varphi(\hat{x}) = \varphi(x)$ follows. So, B' covers B . This completes the proof of Theorem 1.

Using this, we now show:

Theorem 2 (AD + DC). *There is no set $\bar{A} \subseteq \omega^\omega$ and norm $\bar{\varphi} : \bar{A} \rightarrow \omega_2$ such that for every $B \subseteq \bar{A}$, $B \in \Sigma_1^1$, $\{\bar{\varphi}(x) : x \in B\}$ is countable.*

Proof. We suppose not, and fix an $\bar{A}$, $\bar{\varphi}$ to the contrary, i.e., having the strong boundedness property. We also fix A , φ as in Theorem 1. We consider the following game:

$$\begin{array}{ll} \text{I} & x \\ \text{II} & y, z \end{array}$$

Here I plays $x \in \omega^\omega$ and II plays reals $y, z \in \omega^\omega$ (which we can think of as being coded into a single real). If there is an $\alpha < \omega_1$ such that $\mathcal{T}_x \upharpoonright \alpha$ or $\mathcal{T}_y \upharpoonright \alpha$ is not well founded, we let α_0 be the least such. (Here $\mathcal{T}$ is as in the proof of Theorem 1. Recall that $\mathcal{T}_x$ is well founded iff $x \in A$). In this case II wins if $\mathcal{T}_x \upharpoonright \alpha_0$ is not well founded. If both $\mathcal{T}_x, \mathcal{T}_y$ are well founded, so $x, y \in A$, then II wins provided $\varphi(y) > \varphi(x)$, $z \in \bar{A}$, and $\bar{\varphi}(z) = \varphi(y)$.

Claim. II wins the game.

Proof. Using the fact that I has the burden in the above game, standard boundedness arguments show that I can not win. Hence, by AD, II must win.

So, fix a winning strategy, σ , for II in the game. Using again standard boundedness arguments (see [1] for a similar argument) one gets a c.u.b. $C \subseteq \omega_2$ such that if $x \in A$ and $\varphi(x) < \alpha \in C$, then $\varphi(\sigma(x)_0) < \alpha$. We fix such a $C \subseteq \omega_2$. We let $\beta =$ the ω_1^{st} element of C . From the coding lemma and Theorem 1, we let $D \subseteq A$ be Σ_1^1 and such that $\{\varphi(x) : x \in D\} \supseteq \beta$. We

let $\overline{D} = \{z: \text{for some } x \in D, \sigma(x) = (y, z) \text{ for some } y\}$. Clearly $\overline{D} \in \Sigma_1^1$ and $\overline{D} \subseteq \overline{A}$. Also, from the definition of D it is clear that $\{\varphi(z): z \in \overline{D}\}$ is uncountable. This contradiction establishes Theorem 2.

Kechris (see [2]) shows that there is a Π_3^1 set A and norm $\varphi: A \rightarrow \delta_3^1$ satisfying boundedness for Σ_3^1 subsets of A and such that every $B \subseteq A$, $B \in \Sigma_3^1$, can be covered by a $\Sigma_1^1 B' \subseteq A$ in a strong sense, i.e. there is a fixed $\alpha_0 < \delta_3^1$ such that any $B \subseteq A$, $B \in \Sigma_3^1$ can be covered by B' with $\sup\{\alpha: \alpha = \varphi(x) \text{ for some } x \in B', \text{ but } \alpha \neq \varphi(x) \text{ for any } x \in B\} < \alpha_0$. We do not know if this stronger property can be arranged for ω_2 . If so, this would allow us to strengthen Theorem 2 by replacing " Σ_1^1 " by "compact".

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DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125

Current address: Department of Mathematics, University of North Texas, Denton, Texas 76203-3886