DENSITY OF THE LEXICOGRAPHICALLY ORDERED SPACE $\{0, 1\}^\alpha$

W. W. COMFORT

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ABSTRACT. For an infinite cardinal α , the lexicographically ordered space $\{0, 1\}^\alpha$ is denoted $\Lambda(\alpha)$; its density is denoted $d\Lambda(\alpha)$. Completing a computation left unfinished elsewhere, we prove $d\Lambda(\alpha) = 2^\alpha$.

1. BACKGROUND

Juxtaposing in [3] Theorem 11.1(a) of page 29 and 9.1 of page 54, and specializing to cardinal numbers, the theorem emerges that $d\Lambda(\alpha) = \alpha$; this statement appears explicitly in the Introduction to [3]. Correspondence with the author of [3] indicates that Theorem 9.1 of page 54, and hence the statement $d\Lambda(\alpha) = \alpha$, should have been marked in [3] with the symbol $(*)$ (which indicates in [3] that GCH is assumed). The omission of $(*)$ is a typographical error; the author of [3] did not intend or attempt to compute the value of $d\Lambda(\alpha)$ in the absence of GCH.

It is well known and easy to prove for infinite cardinals α that

$$\sum \{2^{|\xi|} : \xi < \alpha, \xi \text{ an ordinal}\} = \sum \{2^\beta : \beta < \alpha, \beta \text{ a cardinal}\}.$$

As usual, this common value is denoted 2^α . The relation $d\Lambda(\alpha) \leq 2^\alpha$, easily derived from Sierpiński's investigation [4] into the "universal" properties of $\Lambda(\alpha)$, is proved explicitly in [1, (5.14(c))].

2. THE NUMBER $\Lambda(\alpha)$

We clarify, amalgamate, harmonize, and complete the results $d\Lambda(\alpha) \leq 2^\alpha$ and [GCH] $d\Lambda(\alpha) = \alpha$ by proving the following theorem (in ZFC, without additional axioms).

Theorem. $d\Lambda(\alpha) = 2^\alpha$.

Proof. $(\leq)$. For $\xi < \alpha$, write $A(\xi) = \{h \in \Lambda(\alpha) : h(\xi) = 0, h(\eta) = 1 \text{ for } \xi < \eta < \alpha\}$, and define $A = \bigcup_{\xi < \alpha} A(\xi)$. From $|A(\xi)| = 2^{|\xi|}$ follows $|A| \leq 2^\alpha$.

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To prove A dense it is enough to show that A meets each nonempty closed interval $[f, g] \subseteq \Lambda(\alpha)$ with $f < g$. Let f and g first differ at $\xi < \alpha$ and define $h \in \Lambda(\alpha)$ by

$$\begin{aligned} h(\eta) &= f(\eta) = g(\eta) && \text{if } \eta < \xi, \\ &= 0 && \text{if } \eta = \xi, \\ &= 1 && \text{if } \xi < \eta < \alpha. \end{aligned}$$

Then $h \in A(\xi) \subseteq A$. Since h and g first differ at ξ , with $g(\xi) = 1$ and $h(\xi) = 0$, we have $h < g$. If $h \neq f$ then h and f first differ at some $\xi' > \xi$ with $h(\xi') = 1$, $f(\xi') = 0$; thus $f = h$ or $f < h$. Thus $h \in [f, g] \cap A$.

($\geq$). We show that $c\Lambda(\alpha)$, the cellularity of $\Lambda(\alpha)$, satisfies $c\Lambda(\alpha) \geq 2^{\alpha}$. (A *cellular family* in a space X is a family of pairwise disjoint nonempty open subsets of X , and cX is defined by

$$cX = \sup\{|\mathcal{J}| : \mathcal{J} \text{ is a cellular family in } X\}.$$

Each cellular family $\mathcal{J}$ in X satisfies $|\mathcal{J}| \leq dX$, so $cX \leq dX$.)

Fix $\xi < \alpha$, and for each $h \in \{0, 1\}^{\xi}$ define $h^{-}, h^{+} \in \Lambda(\alpha)$ by

$$\begin{aligned} h^{-}(\eta) &= h^{+}(\eta) = h(\eta) && \text{for } \eta < \xi, \\ h^{-}(\eta) &= 0, \quad h^{+}(\eta) = 1 && \text{for } \xi \leq \eta < \alpha. \end{aligned}$$

The open intervals (h^{-}, h^{+}) with $h \in \{0, 1\}^{\xi}$ constitute a cellular family, so

$$c\Lambda(\alpha) \geq |\{0, 1\}^{\xi}| = 2^{|\xi|}.$$

3. REMARKS

(1) The relation $d\Lambda(\alpha) = 2^{\alpha}$ agrees with the assertion $d\Lambda(\alpha) = \alpha$ in the presence of GCH, since GCH (easily) implies $\alpha = 2^{\alpha}$.

(2) If GCH fails at γ —that is, if $\gamma < \gamma^{+} < 2^{\gamma}$ —then with $\alpha = \gamma^{+}$ we have

$$d\Lambda(\alpha) = 2^{\alpha} = 2^{\gamma} > \gamma^{+} = \alpha$$

and the relation $d\Lambda(\alpha) = \alpha$ is false.

(3) The inequality $w\Lambda(\alpha) \leq 2^{\alpha}$ is given in [1, (5.14)]. Together with $c\Lambda(\alpha) \geq 2^{\alpha}$ and the general inequality $cX \leq dX \leq wX$, valid for all spaces X , we then have that

$$w\Lambda(\alpha) = d\Lambda(\alpha) = c\Lambda(\alpha) = 2^{\alpha} \text{ for all (infinite) cardinals } \alpha.$$

(4) We note in passing that the cellularity of $\Lambda(\alpha)$ is assumed, in the sense that there is a cellular family $\mathcal{J}$ in $\Lambda(\alpha)$ such that $|\mathcal{J}| = 2^{\alpha}$. To see this, for $\xi < \alpha$ set

$$S(\xi) = \{\eta : \xi + 1 \leq \eta < \xi + \xi\}$$

and for $h \in \{0, 1\}^{S(\xi)}$ define $h^-, h^+ \in \Lambda(\alpha)$ by

$$\begin{aligned} h^-(\eta) &= h^+(\eta) = 1 & \text{if } \eta < \xi, \\ h^-(\eta) &= h^+(\eta) = 0 & \text{if } \eta = \xi, \\ h^-(\eta) &= h^+(\eta) = h(\eta) & \text{if } \eta \in S(\xi), \\ h^-(\eta) &= 0, h^+(\eta) = 1 & \text{if } \xi + \xi \leq \eta < \alpha. \end{aligned}$$

For $\xi < \alpha$ the family $\mathcal{I}(\xi) = \{(h^-, h^+): h \in \{0, 1\}^{S(\xi)}\}$ is a cellular family of cardinality $|\{0, 1\}^{S(\xi)}| = 2^{|S(\xi)|}$. If $f \in \Lambda(\alpha)$ and there is $h \in \{0, 1\}^{S(\xi)}$ such that $f \in (h^-, h^+)$, then $f(\xi) = 0$ and $f(\eta) = 1$ for all $\eta < \xi$. This shows that

$$\left(\bigcup \mathcal{I}(\xi)\right) \cap \left(\bigcup \mathcal{I}(\xi')\right) = \emptyset \quad \text{for } \xi \neq \xi',$$

so $\mathcal{I} = \bigcup_{\xi < \alpha} \mathcal{I}(\xi)$ is a cellular family in $\Lambda(\alpha)$, of cardinality 2^α , as required.

(5) The relation $d\Lambda(\alpha) = 2^\alpha$ will play a rôle in forthcoming work with Dieter Remus [2] concerning the cardinalities of certain classes of topological group topologies.

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REFERENCES

1. W. W. Comfort and S. Negrepointis, *The theory of ultrafilters*, Grundlehren der Math. Wissenschaften, Band 211, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
2. W. W. Comfort and D. Remus, *Long chains of Hausdorff topological group topologies*, (submitted for publication).
3. M. A. Maurice, *Compact ordered spaces*, Mathematical Centre Tracts, no. 6, Mathematisch Centrum, Amsterdam, 1964.
4. W. Sierpiński, *Sur une propriété des ensembles ordonnés*, Fund. Math. **36** (1949), 56–67.

DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457