INDUCTIVE LIMITS OF NORMED ALGEBRAS AND m -CONVEX STRUCTURES

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(Communicated by John B. Conway)

ABSTRACT. In this note, we give a negative answer to a question raised by A. Arosio in 1974. It concerns the multiplicativity of the inductive limit topology associated to a family of normed algebras.

1. INTRODUCTION

We consider the following situation: E is an algebra and $F = (E_i)_{i \in I}$ a family of subalgebras of E such that

- (1) $E = \bigcup E_i$, (2) E_i is a normed algebra for every i , (3) F is a net with continuous injections.

We consider the finest locally convex topology on E making continuous the canonical mappings f_i ([2]). We denote it by t_L . The question is to know if t_L is always m -convex.

Actually the question arose implicitly in 1956 when S. Warner ([9]) considered the finest locally m -convex topology making the f_i continuous.

A. Arosio ([2]) showed that the answer is positive if F is a chain. He did the same if F is countable and E commutative. Using a result of M. Raouyane ([8]), Nacir showed ([7]) that commutativity is not necessary for this result.

By considering the algebra of strongly bounded operators ([1]) and using a result of J. Esterle ([4]) we show that the answer is not always positive in the non-commutative case. The counter-example also contradicts an assertion given without proof in ([5]).

2. PRELIMINARIES

Let E be a complex algebra endowed with a locally convex topology t given by a family $(P_s)_{s \in S}$ of semi-norms, where S is a directed set; (E, t) is said to be:

- (1) A locally convex algebra (l.c.a.) if the product $(x, y) \mapsto x \cdot y$ is separately continuous.

Received by the editors June 20, 1989 and, in revised form, July 24, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46H20; Secondary 46H05.

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 0002-9939/90 \$1.00 + \$.25 per page

- (2) An m -convex algebra (l.m.c.a. [6]) if, for every s , $P_s(xy) \leq P_s(x) \cdot P_s(y)$ for every x, y .
- (3) An A -convex algebra (l.A.c.a. [3]) if, for every s and every x , there exist scalars $M(s, x) > 0$ and $N(s, x) > 0$ such that

$$\begin{aligned} P_s(xy) &\leq M(s, x) \cdot P_s(y) && \text{for every } y, \\ P_s(yx) &\leq N(s, x) \cdot P_s(y) && \text{for every } y. \end{aligned}$$

3. COUNTER EXAMPLE

In the sequel E will designate a Frechet locally convex space. Recall that an operator $T: E \rightarrow E$ is said to be strongly bounded if there exists a neighborhood V of zero such that $T(V)$ is bounded. We also consider the collection $\mathcal{B}_0$ of closed absolutely convex bounded sets of E and $\mathcal{V}$ a fundamental system of absolutely convex neighborhoods of zero such that $\bigcap \{V, V \in \mathcal{V}\} = \{0\}$. We denote by $\mathcal{B}(E)$ the set of strongly bounded operators of E . It is a subalgebra of the algebra of bounded operators of E . If, for every V in $\mathcal{V}$ and every B in $\mathcal{B}_0$ such that $B \subset V$, we put

$$E(V, B) = \{T \in \mathcal{B}(E) \mid \exists \alpha > 0: T(V) \subset \alpha B\} \text{ then}$$

- (1) the $E(V, B)$ are subalgebras the union of which is $\mathcal{B}(E)$.
- (2) Each $E(V, B)$ is endowed with an algebra norm $\|T\| = \sup\{\|T(x)\|_B: x \in V\}$ where $\|\cdot\|_B$ is the gauge of B in the space E_B spanned by B .
- (3) The family $(E(V, B))$ is directed if we put $(V_1, B_1) \leq (V_2, B_2)$, for $V_1 \supset V_2$ and $B_1 \subset rB_2$ for some $r > 0$. And the canonical mappings are continuous.

Now if we put $\Omega_{BV} = \{T \in \mathcal{B}(E) \mid T(B) \subset V\}$, $B \in \mathcal{B}_0$ and $V \in \mathcal{V}$, we get a fundamental system of a locally convex topology t on $\mathcal{B}(E)$. It is Hausdorff and coarser than t_L . Hence t_L is Hausdorff. Let us notice that $A(E) \subset \mathcal{B}(E)$, where $A(E)$ is the algebra of finite rank and continuous operators of E . At last we conclude to the non- m -convexity of t_L by the fact that if E is not normable then the algebra $A(E)$ admits no Hausdorff algebra topology for which the product is continuous ([3, Corollary 2, p. 1159]).

Remark 3.1. In the counter example the product is not continuous on $(\mathcal{B}(E), t_L)$. Also $(\mathcal{B}(E), t_L)$ cannot be a l.A.c.a, since then we could endow it with a Hausdorff m -convex topology by setting

$$q_s(x) = \sup\{P_s(xy): P_s(y) \leq 1\}$$

Actually we first adjoin a unit to $\mathcal{B}(E)$, then consider the restriction of such a topology.

Remark 3.2. The counter example contradicts an affirmation given without proof in ([5, 1.4, p. 57]); it is asserted there that an inductive limit of l.c.a. with continuous products has also a continuous product.

REFERENCES

1. M. Akkar, *Sur la theorie spectrale des operateurs fortement bornés dans les espaces topologiques*, C. R. Acad. Sc. Paris, Série A, **274** (1972), 1805–1808.
2. A. Arosio, *Locally convex inductive limits of normed algebras*, Rend. Sem. Mat. Univ. Padova, **51** (1974), 333–359.
3. A. C. Cochran, R. Keown and C.R. Williams, *On a class of topological algebras*, Pacific J. Math. **34** (1970), 17–25.
4. J. Esterle, *Sur la non metrisabilité de certaines algèbres d'opérateurs*, Rev. Roumaine Math. pures et appl. **XXIV**, no. 8 (1979), 1157–1164.
5. A. Guichardet, *Leçons sur certaines algèbres topologiques*, Gordon and Breach-Dunod, 1967.
6. E. A. Michaël, *Locally multiplicatively convex topological algebras*, Memoirs Amer. Math. Soc., no. 11 Providence, 1952.
7. C. Nacir, *Limites inductives bornologiques convexes et topologiques...*, Thèse de 3ème Cycle, E. N. S. de Rabat, 1986.
8. M. Raouyane, *Algèbres pseudo-Banach normalisées*, Bull. Soc. Math. Belgique **38** (1986), 39–46.
9. S. Warner, *Inductive limits of normed algebras*, Trans. of Amer. Math. Soc. **82** (1956), 190–216.

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