MICHAEL SELECTION THEOREM UNDER WEAK LOWER SEMICONTINUITY ASSUMPTION

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ABSTRACT. We give a continuous selection theorem for convex-valued multifunctions satisfying slightly weaker lower semicontinuity assumptions than those which are adopted in the famous Michael Theorem [4] and in [1].

1. INTRODUCTION

Let $(Y|\cdot|)$ be a normed linear space. Let $B = \{y \in Y|\|y\| \leq 1\}$ denote the closed unit ball and $B^o = \{y \in Y|\|y\| < 1\}$ denote the open unit ball in Y . Denote $\mathcal{N}(Y) = \{C \subset Y|C \neq \emptyset\}$, $\mathcal{C}(Y) = \{C \in \mathcal{N}(Y)|C \text{ is convex}\}$, $\mathcal{CC}(Y) = \{C \in \mathcal{C}(Y)|C \text{ is complete}\}$. For $y \in Y$, $C \in \mathcal{N}(Y)$ set $y + C = \{y + v|v \in C\}$, $rC = \{rv|v \in C\}$ for $r \in \mathbb{R}$.

Let X be a topological space and $F: X \rightarrow \mathcal{N}(Y)$ be a multivalued mapping. A function $f: X \rightarrow Y$ such that $f(x) \in F(x)$ ($f(x) \in F(x) + \varepsilon B^o$) for every $x \in X$ will be called a selection (ε -approximate selection) of F .

A multivalued mapping $F: X \rightarrow \mathcal{N}(Y)$ is called *lower semicontinuous (l.s.c.)* at $x_0 \in X$ iff for every $\varepsilon > 0$ and $z \in F(x_0)$ there exists a neighborhood U_z of x_0 such that

$$z \in \bigcap \{F(x) + \varepsilon B^o | x \in U_z\}.$$

F is called *Hausdorff lower semicontinuous (H.l.s.c.)* at $x_0 \in X$ iff for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that

$$F(x_0) \subset \bigcap \{F(x) + \varepsilon B^o | x \in U\}.$$

Following Deutsch and Kenderov [2], (see also [3]), we call F *almost lower semicontinuous (a.l.s.c.)* at $x_0 \in X$ iff for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that

$$\bigcap \{F(x) + \varepsilon B^o | x \in U\} \neq \emptyset.$$

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Following De Blasi and Myjak [1], we call F *weakly Hausdorff lower semicontinuous* (*w.H.l.s.c.*) at $x_0 \in X$ iff for every $\varepsilon > 0$ and every neighborhood V of x_0 there exist a neighborhood $U \subset V$ of x_0 and a point $x' \in U$ such that

$$F(x') \subset \bigcap \{F(x) + \varepsilon B^o \mid x \in U\}.$$

F is *l.s.c.* (*H.l.s.c.*, *a.l.s.c.*, *w.H.l.s.c.*) if F is *l.s.c.* (resp. *H.l.s.c.*, *a.l.s.c.*, *w.H.l.s.c.*) at each point of X . Clearly if F is *H.l.s.c.*, then F is both *l.s.c.* and *w.H.l.s.c.*, if F is *l.s.c.*, then F is *a.l.s.c.* and if F is *w.H.l.s.c.*, then F is *a.l.s.c.* In general, there are no other implications between these four concepts of lower semicontinuity (see examples below).

Assume now that X is a paracompact space. The famous Michael Theorem [4], tells that if $F: X \rightarrow \mathcal{C}(Y)$ is *l.s.c.*, then F admits a continuous selection. The crux of the matter in Michael's proof is to show that F has a continuous ε -approximate selection f_ε for every $\varepsilon > 0$ and then to show that the multivalued mapping $G: x \rightarrow F(x) \cap (f_\varepsilon(x) + \varepsilon B)$ possesses the same property. Deutsch and Kenderov [2, Theorem 2.4] proved that $F: X \rightarrow \mathcal{C}(Y)$ has a continuous ε -approximate selection iff F is *a.l.s.c.* Unfortunately, for an *a.l.s.c.* F and a continuous ε -approximate selection f_ε of F the multivalued mapping G defined above may not be *a.l.s.c.* Involving an interesting by itself geometric lemma, De Blasi and Myjak [1] showed that if F is *w.H.l.s.c.*, then G is also *w.H.l.s.c.* and therefore Michael's idea of the proof works for weakly Hausdorff lower semicontinuous multifunctions. Since lower semicontinuity does not imply weak Hausdorff lower semicontinuity (and vice versa), Michael's theorem is not a corollary to the De Blasi and Myjak result (and vice versa). However, careful reexamination of the De Blasi and Myjak paper allows us to state Michael's theorem under the assumption, (we call it the weak lower semicontinuity assumption), inspired by their concept of weak Hausdorff lower semicontinuity, but weaker than both lower semicontinuity and weak Hausdorff lower semicontinuity. Namely, we say that $F: X \rightarrow \mathcal{N}(Y)$ is *weakly lower semicontinuous* at $x_0 \in X$, (*w.l.s.c.*), iff for every $\varepsilon > 0$ and every neighborhood V of x_0 there is a point $x' \in V$ such that for every $z \in F(x')$ there is a neighborhood U_z of x_0 such that

$$z \in \bigcap \{F(x) + \varepsilon B^o \mid x \in U_z\}.$$

Straightforward checking shows that if F is *l.s.c.*, then F is *w.l.s.c.* (set $x' = x_0$), if F is *w.H.l.s.c.*, then F is *w.l.s.c.*, and if F is *w.l.s.c.*, then F is *a.l.s.c.* (fix arbitrary V , $x' \in V$ and $z \in F(x')$). It is well-known that if F is compact-valued, then F is *l.s.c.* iff F is *H.l.s.c.* The same proof can be used to show that if F is compact-valued, then F is *w.l.s.c.* iff F satisfies the following *w.H.l.s.c.*-type condition (formally weaker than *w.H.l.s.c.*): for every $x_0 \in X$, every neighborhood V of x_0 and every $\varepsilon > 0$ there exist a point $x' \in V$ and a neighborhood U of x_0 such that

$$F(x') \subset \bigcap \{F(x) + \varepsilon B^o \mid x \in U\}.$$

The following simple examples show that the other converses are not true in general.

Example 1. Let $X = \mathbf{R}$, $Y = \mathbf{R}^2$, $F(x) = \{(t, xt) | t \in (-\infty, \infty)\}$. Then F is l.s.c., hence w.l.s.c., but not w.H.l.s.c. at each $x \in X$.

Example 2. Let $X = Y = \mathbf{R}$. Set $F(x) = [0, n^{-1}]$ for $x = q2^{-n}$, with q odd integer and n a positive integer, $F(x) = [0, 1]$ for the remaining $x \in \mathbf{R}$. Since for every $x_0 \in \mathbf{R}$ there exists a sequence (x_k) converging to x_0 , $x_k = q_k 2^{-n_k}$, $n_k \rightarrow \infty$, it follows that $\bigcap \{F(x) + \varepsilon B^0 | x \in (x_0 - \delta, x_0 + \delta)\} = \varepsilon B^0$ and F is w.H.l.s.c., hence w.l.s.c., but not l.s.c. at every $x_0 \in X$.

Example 3. Let $X = \mathbf{R}$, $Y = \mathbf{R}^2$, $F(x) = \{(t, 0) | t \in [0, +\infty)\}$ for $x = n^{-1}$, $n \in \mathbf{N}$, $F(x) = \{(t, xt) | t \in [-1, +\infty)\}$ for the remaining $x \in \mathbf{R}$. F is neither w.H.l.s.c. nor l.s.c. at $x_0 = 0$, but F is w.l.s.c. at $x_0 = 0$. To verify it, notice that

$$\begin{aligned} (-\varepsilon, \varepsilon(1+\delta)\delta^{-1}) \times \{0\} &\subset \bigcap \{F(x) + \varepsilon B^0 | x \in (-\delta, \delta)\} \subset \\ &\subset (-\varepsilon, \varepsilon(1+\delta)\delta^{-1}) \times (-\varepsilon, \varepsilon) \end{aligned}$$

for every $\varepsilon > 0$, $\delta > 0$.

Example 4. Let $X = Y = \mathbf{R}$. Set $F(x) = [0, 1]$ for x rational and $F(x) = [-1, 0]$ for x irrational. Then F will be a.l.s.c., but not w.l.s.c. at every $x \in X$. Notice that F has the continuous selection $f \equiv 0$.

Actually, by virtue of [2, Lemma 2.3, Theorem 2.7], if X is paracompact, $Y = \mathbf{R}$ and $F: X \rightarrow \mathcal{C}(Y)$ has bounded closed values, then F has a continuous selection iff F is a.l.s.c. The next example shows that the same statement with $Y = \mathbf{R}^2$ is false.

Example 5. Let $X = \mathbf{R}$, $Y = \mathbf{R}^2$, $F(x) = \{(t, xt) | t \in [0, 1]\}$ for x irrational, $F(x) = \{(t, 0) | t \in [0, 1]\}$ for rational $x \neq 0$, $F(0) = (1, 0)$. F is a.l.s.c., but F has no continuous selection.

In §2 we give a repetition of the fundamental lemma of De Blasi and Myjak [1] in slightly modified and sharpened form and we state the continuous selection theorem.

2. CONTINUOUS SELECTIONS OF WEAKLY LOWER SEMICONTINUOUS MULTIFUNCTIONS

The basis for our considerations is constituted by certain stability properties of the intersections of convex sets with the balls. A comprehensive study on such properties will be published by the first author elsewhere. The proof of the lemma below, with $\gamma = 2^{-1}$, has been given in fact in [1]. Although for the proof of our selection theorem, the statement for $\gamma = 2^{-1}$ would suffice, the following general statement is more useful for other applications, (see [6, 7]).

Lemma 1. Let $(Y, \|\cdot\|)$ be a normed linear space and let $\gamma \in [0, 1)$, $\delta \in [0, 1 - \gamma)$. If $E \cap (e + \gamma rB) \neq \emptyset$ for $r \geq 0$, $E \in \mathcal{C}(Y)$, $e \in Y$, then

$$(E + \delta rB) \cap (e + rB + \delta rB) \subset E \cap (e + rB) + M(\gamma, \delta)\delta rB,$$

where $M(\gamma, \delta) = 1 + 2(1 + \gamma)(1 - \gamma - \delta)^{-1}$.

Proof. For $\delta = 0$ or $r = 0$ the statement is obvious, so suppose that $\delta > 0$ and $r > 0$. Without loss of generality we may assume that $e = 0$ (since otherwise we can consider $E - e$ instead of E). Fix an arbitrary $y \in (E + \delta rB) \cap (rB + \delta rB)$ and let $u \in E$ be such that $y - u \in \delta rB$.

Case 1. Suppose $\|y\| \leq (1 - \delta)r$. Then $\|u\| \leq \|y - u\| + \|y\| \leq r$, hence $u \in E \cap rB$. Thus $y = u + (y - u) \in E \cap rB + \delta rB \subset E \cap rB + M(\gamma, \delta)\delta rB$.

Case 2. Suppose $(1 - \delta)r < \|y\|$. Put $\lambda = (\|y\| - (1 - \delta)r)(\|y\| - \gamma r)^{-1}$. Clearly $\lambda > 0$ and

$$\lambda < 2\delta(1 - \gamma - \delta)^{-1},$$

since $\gamma r < (1 - \delta)r < \|y\| < (1 + \delta)r$. Then put $u_\lambda = (1 - \lambda)y$. We have $\|u_\lambda - y\| \leq \lambda\|y\| \leq \lambda(1 + \delta)r$ and since $\lambda\|y\| - \lambda\gamma r = \|y\| - (1 - \delta)r$ we have also

$$\|u_\lambda\| = (1 - \lambda)\|y\| = (1 - \delta - \lambda\gamma)r.$$

Choose $z \in E \cap \gamma rB$ and put $v_\lambda = u_\lambda + \lambda z = (1 - \lambda)y + \lambda z$. Then

$$\|v_\lambda\| \leq (1 - \delta - \lambda\gamma)r + \lambda\gamma r = (1 - \delta)r$$

and $\|v_\lambda - u_\lambda\| \leq \lambda\gamma r$. Take $w = (1 - \lambda)u + \lambda z$. Clearly $w \in E$ and

$$\|w - v_\lambda\| = (1 - \lambda)\|y - u\| \leq (1 - \lambda)\delta r.$$

Therefore

$$\|w\| \leq \|w - v_\lambda\| + \|v_\lambda\| \leq r(1 - \delta + \delta - \lambda\delta) = r(1 - \lambda\delta) \leq r.$$

Thus $w \in E \cap rB$. Moreover,

$$\begin{aligned} \|w - y\| &\leq \|w - v_\lambda\| + \|v_\lambda - u_\lambda\| + \|u_\lambda - y\| \leq r((1 - \lambda)\delta + \lambda\gamma + \lambda(1 + \delta)) \\ &= r(\delta + \lambda(\gamma + 1)) < r\delta(1 + 2(1 + \gamma)(1 - \gamma - \delta)^{-1}). \quad \text{Q.E.D.} \end{aligned}$$

Corollary 1. Let $L > 1$, $d \geq 0$, $\sigma \geq 0$ and $\sigma < 3^{-1}(L - 1)d$ whenever $d > 0$. For every $E \in \mathcal{C}(Y)$ and $e \in Y$ satisfying $E \cap (e + dB) \neq \emptyset$ there follows

$$(E + \sigma B^0) \cap (e + LdB + \sigma B^0) \subset E \cap (e + LdB) + N(L)\sigma B^0,$$

where $N(L) = 1 + 3(L + 1)(L - 1)^{-1}$.

Proof. If $d = 0$ the statement follows trivially, so suppose $d > 0$. If $E \cap (e + dB) \neq \emptyset$, then setting $\gamma = L^{-1}$, $\delta = \sigma(Ld)^{-1}$, $r = Ld$ and applying Lemma 1 we obtain

$$(E + \sigma B) \cap (e + LdB + \sigma B) \subset E \cap (e + LdB) + M(L^{-1}, \sigma(Ld)^{-1})\sigma B.$$

Since $M(L^{-1}, \sigma(Ld)^{-1}) = 1 + 2(L + 1)((L - 1)d - \sigma)^{-1} < N(L)$, the statement follows now immediately. Q.E.D.

The next lemma gives the key of the proof of the selection theorem.

Lemma 2. Assume that $L > 1$. If $F: X \rightarrow \mathcal{C}(Y)$ is w.l.s.c. and $f: X \rightarrow Y$ is continuous, then for every continuous function $d: X \rightarrow [0, \infty)$ such that $F(x) \cap (f(x) + d(x)B) \neq \emptyset$, the multivalued mapping $G: x \rightarrow F(x) \cap (f(x) + Ld(x)B)$ is w.l.s.c.

Proof. Fix arbitrary $x_0 \in X$, a neighborhood V of x_0 and $\varepsilon > 0$. Observe that if $d(x_0) = 0$, then G is l.s.c., hence w.l.s.c. at x_0 . Indeed, we have $G(x_0) = \{f(x_0)\}$; thus choosing a neighborhood U of x_0 such that

$$d(x) < 2^{-1}\varepsilon \quad \text{and} \quad f(x_0) \in f(x) + 2^{-1}\varepsilon B^o \quad \text{for } x \in U,$$

we get

$$f(x_0) \in \bigcap \{F(x) \cap (f(x) + Ld(x)B) + \varepsilon B^o \mid x \in U\},$$

since

$$f(x) \in F(x) \cap (f(x) + d(x)B) + d(x)B \subset F(x) \cap (f(x) + Ld(x)B) + 2^{-1}\varepsilon B^o$$

for every $x \in U$.

So let $x_0 \in X$ be such that $d(x_0) > 0$. Choose a $\sigma > 0$ such that

$$\sigma < \min\{6^{-1}(L-1)d(x_0), N(L)^{-1}\varepsilon\}.$$

Choose a neighborhood W of x_0 , $W \subset V$, such that, for $x, x' \in W$,

$$|d(x) - d(x')| < \delta(2L)^{-1}, \quad 2^{-1}d(x_0) < d(x),$$

$$\|f(x) - f(x')\| < 2^{-1}\sigma.$$

Choose in W a point x' such that for every $z \in F(x')$ there exists a neighborhood U_z of x_0 such that

$$z \in \bigcap \{F(x) + \sigma B^o \mid x \in U_z\}.$$

Now take an arbitrary $w \in F(x') \cap (f(x') + Ld(x')B)$. There exists a neighborhood U_w of x_0 such that $w \in \bigcap \{F(x) + \sigma B^o \mid x \in U_w\}$. Set $W_w = W \cap U_w$. Since

$$\begin{aligned} \|w - f(x)\| &\leq \|w - f(x')\| + \|f(x') - f(x)\| \leq \\ &\leq Ld(x') + 2^{-1}\sigma < Ld(x) + \sigma \end{aligned}$$

for $x \in W_w$, it follows that

$$w \in \bigcap \{(F(x) + \sigma B^o) \cap (f(x) + Ld(x)B + \sigma B^o) \mid x \in W_w\}.$$

Clearly $\sigma < 6^{-1}(L-1)d(x_0) < 3^{-1}(L-1)d(x)$ for $x \in W_w$. Therefore, since $F(x) \cap (f(x) + d(x)B) \neq \emptyset$ for every x , then by Corollary 1

$$(F(x) + \sigma B^o) \cap (f(x) + Ld(x)B + \sigma B^o) \subset F(x) \cap (f(x) + Ld(x)B) + N(L)\sigma B^o,$$

for every $x \in W_w$. Thus

$$w \in \bigcap \{F(x) \cap (f(x) + Ld(x)B) + \varepsilon B^o \mid x \in W_w\}. \quad \text{Q.E.D.}$$

Theorem 1. *If X is a paracompact topological space, Y is a normed linear space and $F: X \rightarrow \mathcal{CC}(Y)$ is w.l.s.c., then F has a continuous selection.*

Proof. Following Michael [4, Theorem 3.2''] (see also [1] or [2, Theorem 2.7]) we need only to construct a sequence of continuous mappings $f_n: X \rightarrow Y$ such that

- (1°) $f_n(x) \in F(x) + 2^{-n}B^o$ for $n = 1, 2, \dots, x \in X$,
- (2°) $f_n(x) \in f_{n-1}(x) + 5 \cdot 2^{-n}B^o$ for $n = 2, 3, \dots, x \in X$.

Indeed, if a sequence (f_n) for which (1°) and (2°) hold is defined, then (f_n) is uniformly Cauchy, hence, by an easy appeal to (1°), (2°) and the completeness of $F(x)$, $x \in X$, uniformly convergent to a continuous mapping which is a continuous selection of F . Since F is a.l.s.c., F has a continuous 2^{-1} -approximate selection f_1 by virtue of [2, Theorem 2.4]. Suppose we have defined $f_1, \dots, f_k$ satisfying (1°) and (2°). Define $G: X \rightarrow \mathcal{C}(Y)$ by

$$G(x) = F(x) \cap (f_k(x) + 2^{-k+1}B).$$

By Lemma 2, (with $L = 2$, $d(x) \equiv 2^{-k}$, $f(x) = f_k(x)$), G is w.l.s.c., hence also a.l.s.c. Therefore G has a continuous 2^{-k-1} -approximate selection f_{k+1} . Then

$$f_{k+1}(x) \in F(x) + 2^{-k-1}B^o$$

and

$$\|f_{k+1}(x) - f_k(x)\| < 2^{-k-1} + 2^{-k+1}. \quad \text{Q.E.D.}$$

Lemma 3. *Let T, X be topological spaces. If $F: X \rightarrow \mathcal{N}(Y)$ is a w.l.s.c. multivalued mapping and $f: T \rightarrow X$ is a continuous and open mapping, then $F \circ f$ is w.l.s.c.*

Proof. Fix $\varepsilon > 0$, $t_0 \in T$. Let W be a neighborhood of t_0 . Since $f(W)$ is a neighborhood of $f(t_0)$, there exists $x' \in f(W)$ such that for every $z \in F(x')$ there exists a neighborhood V_z of $f(t_0)$ such that $z \in \bigcap \{F(x) + \varepsilon B^o \mid x \in V_z\}$. Choose $t' \in W$ such that $x' = f(t')$, and then for every $z \in F(f(t'))$ set $U_z = f^{-1}(V_z)$. Then $z \in \bigcap \{F(f(t)) + \varepsilon B^o \mid t \in U_z\}$. Q.E.D.

Corollary 2. *Let X be a topological space such that $X \times Y$ is paracompact. If $F: X \rightarrow \mathcal{CC}(Y)$ is w.l.s.c. and $m: X \times Y \rightarrow \mathbf{R}$ is a continuous mapping such that $F(x) \cap (y + m(x, y)B) \neq \emptyset$, then for every $L > 1$ there exists a continuous mapping $f_L: X \times Y \rightarrow Y$ such that $f_L(x, y) \in F(x)$ and $\|f_L(x, y) - y\| \leq Lm(x, y)$ for every $(x, y) \in X \times Y$.*

Proof. Define a multivalued mapping $G_L: X \times Y \rightarrow Y$ by

$$G_L(x, y) = F(x) \cap (y + Lm(x, y)B).$$

We need only to show that G_L has a continuous selection. The projections pr_X, pr_Y , defined by $\text{pr}_X(x, y) = x$, $\text{pr}_Y(x, y) = y$, are continuous and open. Hence, in particular, by virtue of Lemma 3, $F \circ \text{pr}_X$ is w.l.s.c. Since

$$G_L(x, y) = ((F \circ \text{pr}_X)(x, y)) \cap (\text{pr}_Y(x, y) + Lm(x, y)B),$$

then by Lemma 2, G_L is w.l.s.c. By Theorem 1, G_L has a continuous selection. Q.E.D.

Remark 1. In order to assure that $X \times Y$ is paracompact it suffices to assume, e.g., that X is paracompact and perfectly normal (see [5]).

Remark 2. One can readily verify (arguing by contradiction) that $F: X \rightarrow \mathcal{N}(Y)$ is w.l.s.c. at x_0 iff for every neighborhood V of x_0 , every $\varepsilon > 0$ and every (not necessarily continuous) selection s of F there exists a neighborhood U of x_0 such that

$$s(V) \cap \bigcap \{F(x) + \varepsilon B^o \mid x \in U\} \neq \emptyset.$$

This observation suggests the following generalization. We say that F is *convex lower semicontinuous (c.l.s.c.)* at x_0 if for every neighborhood V of x_0 , every $\varepsilon > 0$ and every selection s of F there exists a neighborhood U of x_0 such that

$$\overline{\text{co}}s(V) \cap \bigcap \{F(x) + \varepsilon B^o \mid x \in U\} \neq \emptyset.$$

The proof of Lemma 2 may be repeated with slight changes, and consequently Theorem 1 remains valid, if F is allowed to be convex lower semicontinuous instead of weak lower semicontinuous.

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