BOUNDEDNESS OF THE CALDERÓN-ZYGMUND SINGULAR INTEGRAL OPERATORS ON Ba SPACES

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ABSTRACT. A condition for the boundedness of a Calderón-Zygmund operator on a new class of Banach function spaces is studied.

1. INTRODUCTION

A new class of function spaces, denoted by Ba , was introduced by X. X. Ding in [2]. It includes not only the standard Lebesgue integrable spaces, some familiar Orlicz spaces, Orlicz-Sobolev spaces, but also many useful new function spaces. In the past few years, many results have been obtained pertaining to Ba spaces and have been used in both classical analysis and other branches of mathematics. (See, for example, [2-5], [8].)

The main result of this paper is the proof of the boundedness of the Calderón-Zygmund singular integral operator (we will call it C - Z operator), especially the boundedness of the well-known Riesz transformation in Ba spaces.

2. THE BASIC IDEA OF Ba SPACES

Let $B = \{B_1, B_2, \dots, B_m, \dots\}$; $a = \{a_1, a_2, \dots, a_m, \dots\}$. Here, B is a sequence of Banach function spaces, and a is a sequence of non-negative real numbers. Moreover, let $\phi(z) = \sum_{m=1}^{\infty} a_m z^m$ be an entire function. For $f \in \bigcap_{m=1}^{\infty} B_m$, we form a power series as follows

$$I(f, \alpha) = \sum_{m=1}^{\infty} a_m \|f\|_{B_m}^m \alpha^m.$$

Here $\|\cdot\|_{B_m}$ stands for the norm in the B_m space. Now denote by R_f the radius of convergence of the series $I(f, \alpha)$ and by Ba the following measurable function set

$$Ba = \left(f \mid f \in \bigcap_{m=1}^{\infty} B_m, R_f \neq 0 \right).$$

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The set Ba is proved to be a Banach space when we define the norm of an element $f \in \text{Ba}$ by $\|f\|_{\text{Ba}} = \inf_{\alpha > 0} (1/\alpha |I(f, \alpha) \leq 1|)$. For a detailed explanation, see [2].

In this paper we will restrict the Banach spaces $B_m = L_{p_m}(\mathbf{R}^n)$, $m = 1, 2, \dots$, and $p_m > 1$. For simplicity, we will denote $\|\cdot\|_{\text{Ba}}$ by $\|\cdot\|$ and $\|\cdot\|_{L_{p_m}}$ by $\|\cdot\|_{p_m}$.

3. THE BOUNDEDNESS OF THE C-Z OPERATOR

We consider the following C-Z operator T on $L_p(\mathbf{R}^n)$

$$T(f)(x) = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} K(y) f(x-y) dy,$$

where $K(y)$ is a smooth function defined in $\mathbf{R}^n$ except at $x = 0$. $K(y)$ is also positively homogeneous of order $-n$ and its integral mean on the unit sphere S^{n-1} is zero.

It is well known that the operator T is bounded on $L_p(\mathbf{R}^n)$ and

$$(1) \quad \|T(f)\|_p \leq A_p \|f\|_p,$$

where A_p only depends on p and the dimension n . After an easy computation, we see that A_p is continuous in p and

$$(2) \quad A_p = O(1/(p-1)) \text{ as } p \rightarrow 1^+ \text{ and } A_p = O(p) \text{ as } p \rightarrow \infty.$$

See, for example, [1] and [7]. Now we prove

Theorem 1. *If there exists two positive constants α and β such that the Ba spaces satisfy the following condition*

$$(3) \quad 1 < \alpha < p_m < \beta \quad \text{for all } a_m \neq 0,$$

then the C-Z operator T is bounded on the Ba spaces.

Proof. if $\alpha < p < \beta$ then $\exists K > 0$ such that $A_p \leq K$ by the continuity. So if (3) is satisfied, we have $A_{p_m} \leq K$, $m = 1, 2, \dots$. By definition of the Ba norm, we can easily see

$$I(f, 1/\|f\|) = \sum_{m=1}^{\infty} a_m \|f\|_{p_m}^m / \|f\|^m \leq 1, \quad \text{for } \forall f \in \text{Ba},$$

so we have the following estimates

$$\begin{aligned} I(T(f), 1/K\|f\|) &= \sum a_m \|T(f)\|_{p_m}^m / (K\|f\|)^m \\ &\leq \sum a_m A_{p_m}^m \|f\|_{p_m}^m / (k\|f\|)^m \leq 1, \end{aligned}$$

therefore

$$\|T(f)\| = \inf_{c>0} (1/c |I(T(f), c) \leq 1|) \leq K\|f\|, \quad \text{for } f \in \text{Ba}.$$

The theorem has been proved. $\square$

People believe that the sufficient condition (3) for the boundedness of C-Z operator is also necessary. But the proof of this needs a more careful estimate on the constants of the L_p boundedness. We will prove this for a well-known special kind C-Z operator, the Riesz transformation, in the next section.

4. THE BOUNDEDNESS OF R_j IN Ba SPACES

Let $f \in L_p$, $1 < p < \infty$, the n -dimensional Riesz transformation of f is defined by

$$(R_j f)(x) = C_n \int_{\mathbf{R}^n} (x_j - y_j) / |x - y|^{n+1} f(y) dy, \quad j = 1, 2, \dots, n,$$

here

$$C_n = \Gamma\left(\frac{n+1}{2}\right) / \left(\pi\left(\frac{n+1}{2}\right)\right).$$

For any positive number $l > 0$, we define two regions $I(l)$ and $J(l)$ in $\mathbf{R}^n$,

$$I(l) = \{x \mid |x| \leq l, x_i > 0, i = 1, 2, \dots, n\},$$

$$J(l) = \{x \mid |x| > 2l, 0 < \theta_j < \pi/4\},$$

where θ_j is the angle between $\overrightarrow{ox}$ and the j th axis. (See Figure 1 below.) Now we define the function $f_l(x)$ by

$$f_l(x) = \begin{cases} 1 & \text{if } x \in I(l), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma.

$$(4) \quad \|R_j f_l(x)\|_p \geq B(p) \|f_l(x)\|_p \quad \text{for } 1 < p < 2,$$

$B(p)$ is independent of l and $B(p) \rightarrow \infty$ as $p \rightarrow 1^+$, and

$$(5) \quad \|R_j f_l(x)\|_p \geq C(p) \|f_l(x)\|_p \quad \text{for } 2 \leq p < \infty,$$

$C(p)$ is independent of l and $C(p) \rightarrow \infty$ as $p \rightarrow +\infty$.

Note the relationship between $B(p)$, $C(p)$, and the A_p in (2). (Also see [6, 7].)

FIGURE 1

Proof of the lemma.

$$R_j f_l(x) = C_n \int_{\mathbf{R}^n} (x_j - y_j) / |x - y|^{n+1} f_l(y) dy = C_n \int_{I(l)} (x_j - y_j) / |x - y|^{n+1} dy.$$

For $y \in I(l)$ and $x \in J(l)$, $x_j - y_j + l \geq x_j$ and $x_j / |x| = \cos \theta_j \geq \cos \pi/4 = \sqrt{2}/2$, so we have $x_j - y_j \geq (\sqrt{2}/2)|x| - l \geq (\sqrt{2}/2)|x| - |x|/2 = (\sqrt{2}/2 - 1/2)|x|$ and $|x - y| \leq x + l \leq 2|x|$, and so for any $x \in J(l)$

$$\begin{aligned} |R_j f_l(x)| &\geq \left[C_n (\sqrt{2} - 1) / 2^{n+2} \right] \int_{I(l)} 1 / |x|^n dy \\ &= \left[C_n (\sqrt{2} - 1) C l^n / n 2^{n+2} \right] |x|^{-n} \end{aligned}$$

the constant C is defined by

$$(6) \quad \int_{y \in I(l)} dy = C \int_{r=0}^l r^{n-1} dr = C l^n / n.$$

Denoting the constant $C_n (\sqrt{2} - 1) C / n 2^{n+2}$ by C' , we have the following estimates,

$$\begin{aligned} \|R_j f_l(x)\|_p &\geq \left(\int_{x \in J(l)} |R_j f_l(x)|^p dx \right)^{1/p} \\ &\geq C' l^n \left(\int_{x \in J(l)} |x|^{-np} dx \right)^{1/p} \\ &= C' l^n \left(C'' \int_{r=2l}^{\infty} r^{n(1-p)-1} dr \right)^{1/p} \\ &= C' (C'')^{1/p} l^n [(2l)^{n-np} / (np - n)]^{1/p} \\ &= [C' (C'')^{1/p} 2^{(n-np)/p} / (np - n)^{1/p}] l^{n/p} \\ &= [C' (C'')^{1/p} 2^{(n-np)/p} n^{1/p} / C^{1/p} (np - n)^{1/p}] \|f_l\|_p, \end{aligned}$$

where we have used $\|f_l\|_p = (C l^n / n)^{1/p}$ and C'' is a constant independent of l and p . If we let $B(p) = C' (C'')^{1/p} 2^{(n-np)/p} n^{1/p} / C^{1/p} (np - n)^{1/p}$, then it is easy to see that $B(p) \rightarrow \infty$ as $p \rightarrow 1^+$, therefore (4) is proved.

It is easy to see that $\|R_j f_l\|_p / \|f_l\|$ is independent of l . By using a similar estimate, (5) can also be proved. But there is a simple way to see (5) is true. If it were not so, we would have a constant k such that

$$\|R_j f_l(x)\|_{p_m} \leq k \|f_l(x)\|_{p_m}, \quad m = 1, 2, \dots, \quad \text{and } p_m \rightarrow \infty \text{ as } m \rightarrow \infty.$$

Here k is independent of p and l . In particular $(\int_{I(l)} |R_j f_l|^{p_m} dx)^{1/p_m} < k \|f_l(x)\|_{p_m}$. Now letting $m \rightarrow \infty$, we have (note $\|f_l(x)\|_{\infty} = 1$) $|R_j f_l(x)| \leq k$ almost everywhere in $I(l)$, or $|\int_{I(l)} (x_j - y_j) / |x - y|^{n+1} dy| \leq k$. Let $x \rightarrow 0$, we have $\int_{I(l)} y_j / |y|^{n+1} dy \leq k$. This is a contradiction since the integration is divergent. $\square$

We are now ready to prove the main theorem.

Theorem 2. *The Riesz transformation R_j $j = 1, 2, \dots, n$ are bounded operators in Ba spaces if and only if there exist constants α and β such that (3) in §3 is satisfied.*

Proof. The sufficiency has been proved in §3. We now prove the necessity of the condition (3).

We assume there exists a constant A , independent of f , such that

$$(7) \quad \|R_j f\| \leq A \|f\| \quad \text{for all } f \in \text{Ba}.$$

By using (4) in the lemma and (7), we have

$$\begin{aligned} & \sum_{m=1}^{\infty} \left\{ a_m (B(p_m) \|f_l(x)\|_{p_m})^m / (A \|f_l\|)^m \right\} \\ & \leq \sum \{ a_m (\|R_j f_l(x)\|_{p_m})^m / \|R_j f_l\|^m \} = 1. \end{aligned}$$

In particular, $a_m^{1/m} B(p_m) \|f_l\|_{p_m} / (A \|f_l\|) \leq 1$, or

$$(8) \quad a_m^{1/m} \|f_l\|_{p_m} / \|f_l\| \leq A / B(p_m).$$

Note that as $p \rightarrow 1 + 0$, $B(p) \rightarrow \infty$, so if α in (3) does not exist, we may find a $p_{m'} > 1$ such that

$$(9) \quad a_m^{1/m} \|f_l\|_{p_m} / \|f_l\| < \frac{1}{2} \quad \text{for } p_m \in (1, p_{m'}] \text{ and } l \in (0, \infty).$$

Without loss of generality, we assume there is $a_{m''}$ such that $a_{m''} \neq 0$, $p_{m''} < p_{m'}$, and

$$(10) \quad 0 < a_{m''}^{1/m''} \|f_l\|_{p_{m''}} / \|f_l\| < \frac{1}{2}.$$

Now choose l_0 large enough such that $Cl_0^n/n > 1$ and

$$M(Cl_0^n/n)^{1/p_{m'}} < a_{m''}^{1/m''} (Cl_0^n/n)^{1/p_{m''}},$$

where $M = \sup(a_m^{1/m}, m = 1, 2, \dots) < \infty$, C is defined by (6). We can see for any $p_m > p_{m'}$,

$$a_m^{1/m} (Cl_0^n/n)^{1/p_m} \leq M(Cl_0^n/n)^{1/p_{m'}} \leq a_{m''}^{1/m''} (Cl_0^n/n)^{1/p_{m''}},$$

so, by using (10) and the fact $\|f_{l_0}\|_p = (Cl_0^n/n)^{1/p}$, we have

$$(11) \quad a_m^{1/m} \|f_{l_0}\|_{p_m} / \|f_{l_0}\| \leq a_{m''}^{1/m''} \|f_{l_0}\|_{p_{m''}} / \|f_{l_0}\| < 1/2 \quad p_m > p_{m'}.$$

(9) and (11) together give

$$\sum_{m=1}^{\infty} a_m \|f_{l_0}\|_{p_m}^m / \|f_{l_0}\|^m < \sum_{m=1}^{\infty} (1/2^m) = 1,$$

which is a contradiction to the definition of the Ba norm, since we can easily check that $I(f_{l_0}, 1/\|f_{l_0}\|) = 1$.

Now we proceed to prove the existence of the β in the theorem. Using (5) in the lemma and (7), we obtain (by using similar estimates)

$$\sum \{a_m (C(p_m) \|f_l\|_{p_m})^m / (A \|f_l\|)^m\} \leq 1.$$

So $a_m^{1/m} \|f_l\|_{p_m} / \|f_l\| \leq A/C(p_m)$. Note that as $p \rightarrow \infty$, $C(p) \rightarrow \infty$; so if β does not exist, we can find $p_{m'}$ large enough such that

$$(12) \quad a_m^{1/m} \|f_l\|_{p_m} / \|f_l\| < \frac{1}{2} \quad \text{for } p_m \in [p_{m'}, \infty) \text{ and } l \in (0, \infty).$$

Similarly, we may assume $\exists m''$ such that $p_{m''} > p_{m'}$ and

$$(13) \quad 0 < a_{m''}^{1/m''} \|f_l\|_{p_{m''}} / \|f_l\| < \frac{1}{2} \quad \text{for } l \in (0, \infty).$$

Choose l_1 small enough such that $Cl_1^n/n < 1$ and

$$M(Cl_1^n/n)^{1/p_{m'}} < a_{m''}^{1/m''} (Cl_1^n/n)^{1/p_{m''}},$$

then for any $p_m < p_{m'}$,

$$a_m^{1/m} (Cl_1^n/n)^{1/p_m} \leq M(Cl_1^n/n)^{1/p_{m'}} < a_{m''}^{1/m''} (Cl_1^n/n)^{1/p_{m''}}.$$

So, from (13), we have

$$(14) \quad a_m^{1/m} \|f_{l_1}\|_{p_m} / \|f_{l_1}\| \leq a_{m''}^{1/m''} \|f_{l_1}\|_{p_{m''}} / \|f_{l_1}\| < \frac{1}{2} \quad \text{for } p_m < p_{m'}.$$

Equations (12) and (14) give

$$\sum_{m=1}^{\infty} a_m \|f_{l_1}\|_{p_m}^m / \|f_{l_1}\|^m < 1,$$

which is again a contradiction to the definition of the Ba norm, and the theorem has been proved. ■

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