A NOTE ON SOME ELEMENTARY MEASURES OF ALGEBRAIC INDEPENDENCE

ROBERT TUBBS

(Communicated by William Adams)

ABSTRACT. We investigate the algebraic independence of some numbers associated with elliptic functions when one of the numbers is a “Liouville-type” number. Suppose $\wp(z)$ is a Weierstrass elliptic function with algebraic invariants and β is an algebraic number, not belonging to the field of multiplications for $\wp(z)$. We establish the algebraic independence of $\wp(u)$ and $\wp(\beta u)$ (respectively, of u and $\wp(\beta u)$) when $\wp(u)$ (respectively, u) is a “Liouville-type” number. We also give quantitative versions of these results.

In this paper we study the algebraic independence of two numbers associated with Liouville-type numbers and points on an elliptic curve. The motivation for this study comes from two sources: one was to provide an elliptic analogue to a result of N. I. Feldman [3] concerning the algebraic independence of α and α^β when α is “well-approximated” in a certain sense and β is algebraic, irrational; the other was to continue the investigation of the present author in [8] which provided a measure for the transcendence of a special value of a Weierstrass elliptic function.

Some generalizations and extensions of Feldman’s work have already been given. Specifically W. D. Brownawell [1] and Brownawell and M. Waldschmidt [2] have examined the algebraic independence of α , α^β , α^{β^2} when α is “well approximated” and β is cubic, and the present author has provided an elliptic analogue to Brownawell’s result [7]. Additionally M. Laurent [5] has improved the α and α^β result of Feldman, and it is along these lines that we work in the elliptic case.

Let $\wp(z)$ denote a Weierstrass elliptic function with its invariants g_2 and g_3 being algebraic. (Recall that $\wp(z)$ satisfies the differential equation: $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$). The function $\wp(z)$ is doubly periodic and meromorphic. Let $\mathcal{O} \subseteq \mathbb{C}$ denote the ring of endomorphisms of the elliptic curve associated with $\wp(z)$ and let K denote the field of fractions of $\mathcal{O}$.

For an algebraic number α we let $ht(\alpha)$ denote the maximum absolute value of the coefficients of the minimal polynomial of α over $\mathbb{Z}$ and let $\deg \alpha$ denote

Received by the editors October 17, 1988 and, in revised form, June 19, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 11J89; Secondary 11K60.

©1990 American Mathematical Society
 0002-9939/90 \$1.00 + \$.25 per page

the degree of that polynomial. With these conventions we can now make the notation of “well approximated”, referred to above, precise.

Suppose that f is a function defined on the set of nonnegative integers. For the purpose of this paper we say that a complex number ζ is f -approximable if for some positive integer d_0 and an infinite number of positive integers T there exists an algebraic number a_T satisfying

$$\deg a_T \leq d_0, \quad ht(a_T) \leq e^T$$

and

$$0 < |a_T - \zeta| < \exp(-f(T)).$$

We let $\{a_{T_j}\}_{j=1}^\infty$ denote the sequence of algebraic numbers associated with the approximation of ζ .

In each theorem below assume that $\wp(z)$ is a Weierstrass elliptic function with algebraic invariants and that β is a nonzero algebraic number.

Theorem 1. Suppose $\beta \notin K$. If for some $\varepsilon > 2$, $\wp(u)$ is T^ε -approximable, then $\wp(u)$ and $\wp(\beta u)$ are algebraically independent.

Moreover, we have a quantitative version of Theorem 1 which depends more subtly on the sequence $\{a_T\}$ approximating $\wp(u)$.

Theorem 2. Suppose $\beta \notin K$ and that for some $\varepsilon > 2$, $\wp(u)$ is T^ε -approximable. Suppose that $P(x, y)$ is a nonzero integral polynomial of total degree at most d_p and height at most h_p . Put

$$M = \left(\frac{1}{\varepsilon - 1} \right)^{1/\varepsilon} d_0^{14/(\varepsilon-2)} d_p^{8/(\varepsilon-2)} (\log h_p)^{1/(\varepsilon-1)} (\log d_0 + \log d_p + \log \log h_p)^{3/(\varepsilon-2)}.$$

Then there exists a constant S_1 , depending only on d_0 , β , $\wp$, and ε , such that if $\min\{d_p, h_p\} \geq S_1$ one has

$$|P(\wp(u), \wp(\beta u))| > \exp(-t^\varepsilon M^\varepsilon),$$

where

$$t = \frac{T_{k+1}}{T_k}, \quad k = \max\{j: T_j \leq M\}.$$

We also consider the values u and $\wp(\beta u)$ where u is “well approximated” by algebraic numbers. This is the object of the next two theorems.

Theorem 3. Suppose $\beta \neq 0$. If for some $\varepsilon > 3$, u is T^ε -approximable, then u and $\wp(\beta u)$ are algebraically independent.

As did Theorem 1, Theorem 3 also has a quantitative version.

Theorem 4. Suppose $\beta \neq 0$ and that for some $\varepsilon > 3$, u is T^ε -approximable. Suppose that $P(x, y)$ is a nonzero integral polynomial of total degree at most d_p and height at most h_p . Put

$$M_1 = \left(\frac{1}{\varepsilon - 2} \right)^{1/\varepsilon} d_0^{10/(\varepsilon-3)} d_p^{6/(\varepsilon-3)} (\log h_p)^{1/(\varepsilon-2)}.$$

Then there exists a constant S_2 , depending only on d_0 , β , $\wp$, and ε , such that if $\min\{d_p, h_p\} \geq S_2$, one has

$$|P(u, \wp(\beta u))| > \exp(-t_1^\varepsilon M_1^\varepsilon),$$

where

$$t_1 = \frac{T_{k+1}}{T_k}, \quad k = \max\{j: T_j \leq M_1\}.$$

The proofs of these theorems are based on three lemmas. The first of these is very elementary, the second is still rather basic, but the third is a very deep result due to Patrice Philippon and Michel Waldschmidt.

Lemma 1. *There exist constants k_1 and k_2 depending on u and $\wp(z)$ such that if $|a - \wp(u)| < k_1$, then there exists $w \in \mathbb{C}$ with $\wp(w) = a$ and $|w - u| < k_2|\wp(w) - \wp(u)|$, provided $\wp'(u) \neq 0$.*

Proof. Elementary, using the existence of a locally (at u) analytic inverse of $\wp(z)$.

As hinted at above the next lemma is not quite as elementary.

Lemma 2. *Suppose $P(X) \in \mathbb{Z}[X]$ is a nonzero polynomial with $d = \deg P$ and $h = ht P$. Let ζ_i ($1 \leq i \leq k$) denote the distinct roots of P with multiplicities m_i . Then for any $\theta \in \mathbb{C}$*

$$\min_{1 \leq i \leq k} |\theta - \zeta_i|^{m_i} \leq (2d) \cdot 2^d \cdot (1+d)^{2d} e^{2d^2} h^{2d} |P(\theta)|.$$

Proof. See Lemma 5.3.7 of [9].

To state our third lemma we need a bit of notation. Suppose that $u_1, \dots, u_n$ are complex numbers such that each of $\wp(u_1), \dots, \wp(u_n)$ is algebraic, and suppose further that $\beta_0, \beta_1, \dots, \beta_n$ are arbitrary algebraic numbers. Put $L = \mathbb{Q}(g_2, g_3, \beta_0, \dots, \beta_n, \wp(u_1), \dots, \wp(u_n))$ and $D = [L: \mathbb{Q}]$. Moreover let $B, E, V_1, \dots, V_n$ be real numbers satisfying

$$\begin{aligned} \log V_i &\geq \min\{\log ht(\wp(u_i)), |u_i|^2/D\}, \quad 1 \leq i \leq n, \\ B &\geq \max\{D \log V_1, \dots, D \log V_n, ht(\beta_0), \dots, ht(\beta_n)\} \\ E &\leq \min\{B^D, \min_{1 \leq i \leq n} \{e(D \log V_i)^{1/2}/|u_i|\}\}. \end{aligned}$$

Lemma 3. *Let $\Lambda = \beta_0 + \beta_1 u_1 + \dots + \beta_n u_n$. There exists a constant $C_3 > 0$ depending only on n , g_2 , and g_3 such that if $\Lambda \neq 0$, we have*

$$|\Lambda| > \exp\left(-C_3 D^{2n+2} (\log B)^{n+1} \left(\prod_{i=1}^n \log V_i\right) \log(DE) (\log E)^{-2n-1}\right).$$

Proof. This is Théorème 1.1 of [6] with $d_1 = 0$ and $h(x) = \log ht(x)$.

We present here a complete proof of Theorem 2 and then only indicate changes which are necessary to establish Theorem 4. Each of these proofs is a

variant of the proof of Théorème 1 of [5]. Clearly Theorem 1 and Theorem 3 follow from these results. The lower case constants $c_1, \dots, c_{13}$ which appear below are effective and depend at most on u , β , and $\wp(z)$.

Proof of Theorem 2. We begin by noting that $\wp'(\beta u) \neq 0$; or, equivalently, $E(\wp(\beta u)) \neq 0$ where $E(X) = 4X^3 - g_2X - g_3$. To obtain this let $R_T(X)$ denote the minimal polynomial of a_T over $\mathbf{Q}$ where degree $R_T \leq d_0$ and height $R_T \leq e^T$. Since $R_T(a_T) = 0$ we obtain

$$|R_T(\wp(u))| = |R_T(\wp(u)) - R_T(a_T)| \leq e^T(1 + d_0)^2(1 + |\wp(u)|)^{d_0}e^{-T^\varepsilon}.$$

If $E(\wp(\beta u)) = 0$, then $\wp(\beta u) \in \overline{\mathbf{Q}}$ and from $\beta \notin K$ with $\wp(u) = \wp(\beta^{-1}\beta u)$ we deduce from the Theorem of [8] that

$$|R_T(\wp(u))| \geq e^{-c_1 d_0^6 (d_0 + T)^2 (\log(d_0 + T))^{14}}.$$

Since $\varepsilon > 2$, these bounds are incompatible as T approaches infinity.

Suppose that $P(x, y) \in \mathbf{Z}[x, y]$ is a nonzero polynomial of total degree at most d_p and height at most h_p with

$$|P(\wp(u), \wp(\beta u))| \leq \exp(-t^\varepsilon M^\varepsilon),$$

with t and M as in the statement of the theorem.

Take S_1 sufficiently large so that $M \geq T_1$. Then by our choice of $k = \max\{j: T_j \leq M\}$ we have

$$T_k \leq M \leq T_{k+1} \leq tM.$$

Put $T = T_{k+1}$ and let $a_T = \alpha_1, \dots, \alpha_\delta$ denote the conjugates of a_T (here δ denotes the degree of a_T so that $\delta \leq d_0$). Also, let q denote a denominator for a_T , so that $|q| < e^T$.

Then

$$(1) \quad Q(y) = q^{\delta d_p} \prod_{j=1}^{\delta} P(\alpha_j, y)$$

is a nonzero integral polynomial with

$$(2) \quad \begin{aligned} ht(Q) &\leq e^{3T\delta d_p} h_p^\delta (1 + d_p)^{2\delta}, \\ \deg Q &\leq \delta \cdot d_p. \end{aligned}$$

Our immediate goal is to estimate $|Q(\wp(\beta u))|$, which requires that we estimate $|P(\alpha, \wp(\beta u))|$ for each $j = 1, \dots, \delta$.

For $j = 1$ we have

$$(3) \quad \begin{aligned} |P(\alpha_1, \wp(\beta u))| &\leq |P(\alpha_1, \wp(\beta u)) - P(\wp(u), \wp(\beta u))| + |P(\wp(u), \wp(\beta u))| \\ &\leq h_p(1 + d_p)^3(1 + |\wp(u)|)^{d_p} \max\{1, |\wp(\beta u)|\}^{d_p} |a_T - \wp(u)| \\ &\quad + e^{-t^\varepsilon M^\varepsilon} \\ &\leq c_2^{d_p} h_p(1 + d_p)^3 e^{-T^\varepsilon}, \end{aligned}$$

by our choice of T . For the other choices of j we have the general estimate

$$|P(\alpha_j, \wp(\beta u))| \leq h_p(1 + d_p)^2 \max\{1, |\alpha_j|\}^d \max\{1, |\wp(\beta u)|\}^{d_p}.$$

If we then apply the estimate $\max\{1, |\alpha_j|\} \leq ht(a_T) + 1 \leq e^{2T}$, we deduce that for $j = 2, \dots, \delta$

$$(4) \quad |P(\alpha_j, \wp(\beta u))| \leq c_3^{d_p} h_p(1 + d_p)^2 e^{2Td_p}.$$

Combining (3) and (4) we deduce that

$$\begin{aligned} |Q(\wp(\beta u))| &\leq c_4^{\delta d_p} h_p^\delta (1 + d_p)^{2\delta+1} e^{3T\delta d_p - T^\epsilon} \\ &\leq c_4^{\delta \cdot d_p} h_p^\delta (1 + d_p)^{2\delta+1} e^{-T^\epsilon/2}, \end{aligned}$$

provided S_1 is sufficiently large.

We next choose a root γ of $Q(y)$ which is close to $\wp(\beta u)$. By Lemma 2 there exists such a γ , of multiplicity m , with

$$(5) \quad \begin{aligned} |\gamma - \wp(\beta u)|^m &\leq e^{7d_p^2 \delta^2} ht(Q)^{2d_p \delta} \cdot |Q(\wp(\beta u))| \\ &\leq e^{c_5 \delta^2 d_p (Td_p + \log h_p) - T^\epsilon/2}. \end{aligned}$$

Since $1 \leq m \leq d_p \delta$, we deduce that

$$\begin{aligned} |\gamma - \wp(\beta u)| &\leq e^{c_5(T\delta d_p + \delta \log h_p) - T^\epsilon d_p^{-1} \delta^{-1}/2} \\ &\leq e^{-T^\epsilon d_p^{-1} \delta^{-1}/3}, \end{aligned}$$

by our choice of T , provided S_1 is sufficiently large.

We recall that $\wp'(\beta u) \neq 0$ and then apply Lemma 1 to choose $w_T \in \mathbb{C}$ such that $\wp(w_T) = \gamma$ and

$$|w_T - \beta u| < k_2 e^{-T^\epsilon d_p^{-1} \delta^{-1}/3}.$$

Moreover, $\wp(u)$ is transcendental by a simple Liouville estimate; therefore $\wp'(u) \neq 0$ and by Lemma 1 there exists $v_T \in \mathbb{C}$ such that $\wp(v_T) = a_T$ and

$$|v_T - u| < k_2 e^{-T^\epsilon}.$$

Therefore,

$$(6) \quad |w_T - \beta v_T| \leq c_6 e^{-T^\epsilon d_p^{-1} \delta^{-1}/3}.$$

Note that each of $\wp(w_T)$ and $\wp(v_T)$ is algebraic with

$$\deg \wp(w_T) \leq \deg Q \leq \delta \cdot d_p, \quad \deg \wp(v_T) = \deg a_T = \delta$$

and

$$ht\wp(w_T) \leq 4^{\delta d_p} htQ, \quad ht\wp(v_T) \leq e^T.$$

If $w_T - \beta v_T = 0$, then $\wp(\beta v_T) \in \overline{\mathbb{Q}}$, which along with all of g_2, g_3 , and $\wp(v_T) = a_T$ being algebraic contradicts the Schneider–Lang theorem, Theorem 1, p. 21, [4]. Therefore $w_T - \beta v_T \neq 0$ and we apply Lemma 3 with the

following choices:

$$\begin{aligned} L &= \mathbf{Q}(g_2, g_3, \beta, \wp(v_T), \wp(w_T)), \quad D = [L: \mathbf{Q}], \\ \log V_1 &= \log ht\gamma, \quad \log V_2 = T, \quad B = D \max\{T, \log ht\gamma\}, \\ E &= \min\{e(D \log ht\gamma)^{1/2}/|w_T|, \quad e(DT)^{1/2}/|v_T|\} \end{aligned}$$

to find that

$$|w_T - \beta v_T| > \exp(-C_3 D^6 (\log B)^3 (\log htQ) T (\log D) (\log E)^{-4}).$$

Recalling our estimate for $\deg Q$ and htQ , (2) above and $D \leq \delta^2 d_p$, we obtain

$$|w_T - \beta v_T| > \exp(-c_7 \delta^{13} d_p^6 T (Td_p + \log h_p) (\log T)^3).$$

This last inequality contradicts (6), thanks to our choice of $T \geq M$, provided we take S_1 to be sufficiently large. Thus Theorem 2 is established.

As we mentioned above the proof of Theorem 4 follows the same outline as the proof of Theorem 2, hence we only indicate where the proofs differ.

Proof of Theorem 4. Suppose that $P(X, Y) \in \mathbf{Z}[X, Y]$ is a nonzero polynomial of total degree at most d_p and height at most h_p with

$$|P(u, \wp(\beta u))| \leq \exp(-t_1^\epsilon M_1^\epsilon),$$

here with t_1 and M_1 as in the statement of Theorem 4. As in the proof of Theorem 2 above, take S_2 sufficiently large so that there exists a_T approximating u with

$$M_1 \leq T \leq t_1 M_1.$$

Let $a_T = \alpha_1, \dots, \alpha_\delta$ denote the conjugates of a_T .

Define $Q(y)$ as in (1) in the proof of Theorem 2, and note that $ht(Q)$ and $\deg Q$ satisfy the same estimates as before. To estimate $|Q(\wp(\beta u))|$ we now have

$$|P(\alpha_1, \wp(\beta u))| \leq c_8^{d_p} h_p (1 + d_p)^3 e^{-T^\epsilon}$$

and for $j = 2, \dots, \delta$,

$$|P(\alpha_j, \wp(\beta u))| \leq c_9^{d_p} h_p (1 + d_p)^2 e^{2Td_p}.$$

Combining these estimates we obtain

$$|Q(\wp(\beta u))| \leq c_{10}^{\delta \cdot d_p} h_p^\delta (1 + d_p)^{2\delta+1} e^{-T^\epsilon/2},$$

provided S_2 is sufficiently large.

We apply Lemma 2 to choose a root γ of $Q(y)$ of multiplicity m' such that (5) holds with m' replacing m . Therefore, our choice of T yields

$$|\gamma - \wp(\beta u)| < e^{-T^\epsilon d_p^{-1} \delta^{-1/3}},$$

provided S_2 is sufficiently large. We remark that $\wp'(\beta u) \neq 0$, for otherwise $\wp(\beta u) \in \overline{\mathbf{Q}}$ and we apply Lemma 3 to $\beta_0 = \beta a_T$, $u_1 = \beta u$ to obtain

$$|\beta a_T - \beta u| > \exp(-c_{12} \delta^4).$$

However this contradicts the upper bound

$$|\beta a_T - \beta u| < |\beta| e^{-T^\epsilon},$$

provided S_2 is taken sufficiently large, so that T is sufficiently large. Using Lemma 1 we may then choose $w_T \in \mathbb{C}$ such that $\wp(w_T) = \gamma$ and

$$|w_T - \beta u| < k_2 e^{-T^\epsilon d_p^{-1} \delta^{-1/3}}.$$

Then

$$(7) \quad |w_T - \beta a_T| < c_{11} e^{-T^\epsilon d_p^{-1} \delta^{-1/3}}.$$

Note that $\wp(w_T)$ is algebraic with

$$\deg \wp(w_T) \leq \deg Q, \quad ht \wp(w_T) \leq 4^{\delta dp} ht Q$$

and βa_T is algebraic with

$$\deg \beta a_T \leq (\deg \beta) d_0, \quad ht(\beta a_T) \leq e^{T \deg \beta} ht(\beta)^{d_0}.$$

Moreover, if $w_T = \beta a_T$, then w_T and $\wp(w_T)$ are simultaneously algebraic, which together with g_2 and g_3 being algebraic contradicts the Schneider–Lang theorem.

Therefore $w_T - \beta a_T \neq 0$ and we apply Lemma 3 with

$$\begin{aligned} L &= \mathbf{Q}(g_2, g_3, \beta a_T, \wp(w_T)), & D &= [L: \mathbf{Q}], \\ \log V_1 &= \log ht \gamma, & B &= \max\{D \log V_1, e^T\}, \\ E &= \min\{B^D, e^{(D \log V_1)^{1/2}} / |w_T|\} \end{aligned}$$

to obtain

$$|w_T - \beta a_T| \geq \exp(-c_{13} \delta^9 d_p^4 T^2 (T d_p + \log h_p)).$$

If S_2 is taken to be sufficiently large, this last inequality contradicts (7). This established Theorem 4.

REFERENCES

1. W. D. Brownawell, *On the Gelfond-Feldman measure of algebraic independence*, *Compositio Math.* **38** (3) (1979), 355–368.
2. W. D. Brownawell and M. Waldschmidt, *The algebraic independence of certain numbers to algebraic powers*, *Acta Arith.* **32** (1977), 63–71.
3. N. I. Feldman, *Arithmetic properties of the solutions of a transcendental equation*, *Vestnik Moskov. Univ. Ser. I*, **1** (1964), 13–20.
4. S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, Reading, MA, 1966.
5. M. Laurent, *Indépendance Algébrique de nombres de Liouville à des puissances algébriques*, These 3^e Cycle, L'Université Pierre et Marie Curie, October 1977.
6. P. Philippon and M. Waldschmidt, *Formes lineaires de logarithmes elliptiques et mesures de transcendance*, in *Number Theory Proceedings of International Number Theory Conference*, Quebec, de Gruyter, Berlin, 1989.

7. R. Tubbs, *A diophantine problem on elliptic curves*, Trans. Amer. Math. Soc. **309** (1988), 325–338.
8. ———, *A transcendence measure for some special values of elliptic functions*, Proc. Amer. Math. Soc. (2) **88** (1983), 189–196.
9. M. Waldschmidt, *Nombres transcendants*, Lecture Notes in Math., vol. 402, Springer-Verlag, New York, 1974.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80309