CANCELLATION COUNTEREXAMPLES IN KRULL DIMENSION 1

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ABSTRACT. We show that direct-sum cancellation can fail for modules in a single genus, over commutative noetherian rings of dimension 1. The example provides an application of noncommutative methods (including “wild representation type”) to commutative ring theory.

Let Λ be a module-finite algebra over a commutative ring R . We say that a Λ -module N is in the *genus of M* , and write $N \in \text{gen}(M)$, if $N_{\mathfrak{m}} \cong M_{\mathfrak{m}}$ (as $\Lambda_{\mathfrak{m}}$ -modules) for all maximal ideals $\mathfrak{m}$ of R . (In this paper “module” means “finitely generated left module”, unless the contrary is stated.) We say that *cancellation holds in $\text{gen}(M)$* if

$$(0.1) \quad L \oplus X \cong N \oplus X \Rightarrow L \cong N \quad \text{when } L, N, X \in \text{gen}(M).$$

In [GL '90] it was shown that, if Λ is a commutative noetherian ring of Krull dimension 1 without nilpotent elements, then cancellation holds in every genus of Λ -modules. We give an example showing that this can fail if Λ has nilpotent elements. In our example, the abelian groups $(\Lambda, +)$ and $(M, +)$ are free of finite rank; and the example shows that such failure of cancellation is quite common in this situation. The example makes use of the notion of wild representation type to transfer a noncommutative example of Swan (involving totally definite quaternion algebras) to this commutative situation.

1. *Notation.* Let Λ be a commutative $\mathbf{Z}$ -algebra such that the abelian group $(\Lambda, +)$ is free of finite rank. Suppose that some maximal ideal $\mathfrak{m}$ of the artinian ring $A = \mathbf{Q}\Lambda$ requires at least three generators and $A/\mathfrak{m} \cong \mathbf{Q}$. Such a Λ has Krull dimension 1 because it is integral over $\mathbf{Z}$.

For example, we can take $\Lambda = \mathbf{Z}[x, y, z]/\langle x, y, z \rangle^2$.

2. **Theorem.** *There is a Λ -module M such that $(M, +)$ is free of finite rank and cancellation does not hold in $\text{gen}(M)$.*

Proof. The $\mathbf{Q}$ -algebra $A = \mathbf{Q}\Lambda$ has *wild representation type*, by which we mean that there is a representation equivalence $\Phi: \mathcal{C} \rightarrow \mathcal{F}$, from a full subcategory

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$\mathcal{C}$ of the category of left A -modules, to the category $\mathcal{F}$ of all left modules over the free algebra $\mathbf{Q}\langle s, t \rangle$ in two noncommuting indeterminates.

The definition of *representation equivalence* $\Phi: \mathcal{C} \rightarrow \mathcal{F}$ is not completely standard. We define it to be an additive functor Φ such that: (i) Every isomorphism class in $\mathcal{F}$ occurs in the image of Φ ; (ii) $\Phi(M) \cong \Phi(N)$ if and only if $M \cong N$; and (iii) Φ is a surjection on hom groups. We will use the fact that, in this particular situation, Φ can be chosen to have the following additional property: (iv) M is finite-dimensional if $\Phi(M)$ is.

The existence of Φ follows from a slight modification of [W '78, Theorem 1], whose proof we sketch in Lemma 3 below. The existence of Φ might also follow from an earlier theorem of Drozd [D '72], but we do not know whether Drozd's functor has property (iii) in our definition of "representation equivalence", which will be needed below.

Every finite-dimensional $\mathbf{Q}$ -algebra H can occur as the endomorphism ring of some finite-dimensional $\mathbf{Q}\langle s, t \rangle$ -module. (This is true for any field in place of $\mathbf{Q}$.) [B '74, p. 134; Ga '75, pp. 148–149]

Let H be a $\mathbf{Q}$ -algebra such that direct-sum cancellation fails for projective modules Γ -modules in $\text{gen}(\Gamma)$ over some maximal $\mathbf{Z}$ -order Γ in H . Such an H (a totally definite quaternion algebra) exists by [S '62]. Then we have $H = E(V)$, the endomorphism ring of some finite-dimensional $\mathbf{Q}\langle s, t \rangle$ -module V . Our representation equivalence Φ yields a finite-dimensional A -module M^* such that $V \cong \Phi(M^*)$.

Let M be any (necessarily $\mathbf{Z}$ -free) finitely generated Λ -submodule of M^* such that $\mathbf{Q}M = M^*$. We claim that cancellation fails in the genus of M .

First of all, V is an indecomposable $\mathbf{Q}\langle s, t \rangle$ -module, since $E(V)$ is a division ring. Since Φ is a representation equivalence, M^* is therefore an indecomposable A -module. Since Φ is a surjection on hom groups, we have $\Phi(E(M^*)) = H$, and this is a homomorphism of $\mathbf{Q}$ -algebras (since Φ is an additive functor). Moreover, since the A -module M^* is indecomposable and of finite length, its endomorphism ring is a local ring, so we have

$$(2.1) \quad \ker(\Phi: E(M^*) \rightarrow H) = \text{rad } E(M^*).$$

Now consider $E(M)$. We have $\mathbf{Q} \cdot E(M) = E(\mathbf{Q}M) = E(M^*)$. Since Φ is a $\mathbf{Q}$ -algebra homomorphism on $E(M^*)$, we see that $\Phi(E(M))$ is a $\mathbf{Z}$ -order in H . Since cancellation fails in $\text{gen}(\Gamma)$ for some maximal $\mathbf{Z}$ -order Γ in H , it fails in $\text{gen}(\Lambda)$ for every $\mathbf{Z}$ -order Λ in H (by Lemma 4, below). Therefore cancellation fails for $\Phi(E(M))$ -modules in $\text{gen}(\Phi(E(M)))$.

By (2.1), and the fact that $\Phi(E(M))$ is an integral domain, the kernel of $E(M) \rightarrow \Phi(E(M))$ equals the nilradical of $E(M)$. So, by the proof of [G '84, 3.5], failure of cancellation in $\text{gen}(\Phi(E(M)))$ lifts to failure in $\text{gen}(E(M))$ and then to $\text{gen}(M)$, as desired. $\square$

3. Lemma. *Let A be a commutative artinian ring with a maximal ideal $\mathfrak{m}$ that requires at least three generators, and let $k = A/\mathfrak{m}$. Then there is a representation equivalence Φ from a full subcategory of $\text{mod } A$ to the category of all left modules over the free noncommutative k -algebra $k\langle s, t \rangle$. Moreover, M has finite length if $\Phi(M)$ is finite-dimensional.*

Proof. If H is an ideal of A , then every A/H -module is also an A -module. So it suffices to prove the lemma with A/H in place of A , for some convenient H . We use this to achieve some simplifications. Since A is artinian and commutative, it is a direct product of local rings. So we can assume that A is local with maximal ideal $\mathfrak{m}$. After replacing A by $A/\mathfrak{m}^2$ we can suppose that $\mathfrak{m}^2 = 0$. The k -dimension of $\mathfrak{m}$ is now the minimal number of generators of $\mathfrak{m}$. So after passing to another homomorphic image of $\mathfrak{m}$ we can suppose that this dimension is exactly 3.

Let a, b, c be a k -basis of $\mathfrak{m}$, hence a set of generators for the ideal $\mathfrak{m}$. If M is any A -module, multiplication by a, b, c , respectively, induces k -linear transformations α, β, γ from $M/\mathfrak{m}M$ to $\mathfrak{m}M$. We say that M is *a-translatable* if α is an isomorphism.

If M is *a-translatable*, we define linear transformations ("translations") $s, t: M/\mathfrak{m}M \rightarrow M/\mathfrak{m}M$ by $s = \alpha^{-1}\beta$ and $t = \alpha^{-1}\gamma$ (β or γ first, then α^{-1}). This gives $M/\mathfrak{m}M$ a natural left $k\langle s, t \rangle$ -module structure. If M and N are *a-translatable* R -modules, and $f: M \rightarrow N$ is a homomorphism, then f induces a $k\langle s, t \rangle$ -module homomorphism $M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$. We therefore have a functor Φ from the category of *a-translatable* A -modules to the category of left $k\langle s, t \rangle$ -modules.

In the original theorem in [W '78], $\mathfrak{m}$ had k -dimension 2, and the present role of $k\langle s, t \rangle$ was played by the commutative polynomial ring $k[t]$.

The proof of the present lemma is identical to that of [W '78], with the following change. Every submodule K of a free $k[t]$ -module F is again free, because $k[t]$ is a principal ideal domain. (This is used on p. 453, first full paragraph.) Although $k\langle s, t \rangle$ is not a principal ideal domain, it is a free ideal ring, and over any such ring, submodules of free modules are free [C '71, p. 80]. $\square$

4. Lemma. *Suppose that cancellation fails in $\text{gen}(\Gamma)$, for some maximal $\mathbf{Z}$ -order Γ is some $\mathbf{Q}$ -division algebra H . Then it fails in $\text{gen}(\Lambda)$ for every $\mathbf{Z}$ -order Λ in H .*

Proof. Since all maximal orders in H are Morita equivalent to each other [R '75, (2.17)], cancellation in $\text{gen}(\Gamma')$ fails over all maximal orders Γ' in H . Since, in $\mathbf{Q}$ -division algebras, every order is contained in a maximal order, it suffices to show that this failure of cancellation passes down from maximal orders to all orders contained in them. This is a theorem of Fröhlich [F '75], and is also a special case of [GL '89, 5.4]. $\square$

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