A THOMSON'S PRINCIPLE FOR INFINITE, NONLINEAR RESISTIVE NETWORKS

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ABSTRACT. Suppose Γ is an infinite resistive electrical network with resistors of the form (2). We prove an existence and uniqueness theorem for the current generated by external current sources by establishing the analogue of Thomson's principle in suitable modular sequence spaces.

1. INTRODUCTION

An infinite resistive electrical network consists of an infinite, locally finite connected graph Γ whose edges B are assigned a relation

$$(1) \quad F_B(v_B, i_B) = 0$$

between the voltage v_B and the current i_B flowing in B . We shall assume that the network is energized by external current sources represented by a 0-chain. The voltage and current distributions in the network obey the classical Kirchhoff's laws.

When the relations (1) take the form: $v_B = i_B r_B$ (where r_B is the resistance of the edge B), the network is called linear or ohmic. The theory of infinite linear networks was founded by H. Flanders [F], who proved that, when the number of energy sources is finite, there exists a current of finite energy flowing in the network, and that such a current is unique provided it satisfies the extra condition of being a limit of finite currents. The case of infinitely many sources was considered by Zemanian [Z1, Theorem 3.3], always under the assumption of finite total power.

In this paper we study nonlinear networks where the relations (1) take the form

$$(2) \quad v_B = \text{sign}(i_B) p_B(|i_B|),$$

where p_B is a continuous strictly increasing function such that $p_B(0) = 0$ (and satisfying a technical condition, see (4b)).

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Finite nonlinear networks of this kind were considered in the pioneering papers by Duffin [Du] and Minty [M]. Infinite nonlinear (not necessarily resistive) networks were studied in great generality by Dolezal [D1, D2] and by Dolezal and Zemanian [D-Z] in the context of Hilbert spaces (and total finite power).

As noted by Zemanian, Kirchhoff's laws alone, coupled with the relationships between current and voltage distributions given by the elements of the network, are in general too weak to yield a unique solution (cf. the papers [Z2, Z3, Z4]; see also, in this context, §4 in Soardi and Woess [S-W]). In the papers mentioned in the previous paragraphs the natural approach is taken of restricting the class of admissible solutions to the class of all currents of finite energy. Accordingly, the right mathematical setting is provided by the Hilbert space. However, in nonlinear resistive networks where the resistors have the form (2), the finite energy restriction would impose too strong conditions on the functions p_B .

Our starting point is the illuminating discussion in Thomassen's paper [T] of currents in finite monotone networks. The role of the energy of a 1-chain $I = \sum_{B \in E(\Gamma)} i_B B$ is played by the functional

$$(3) \quad W(I) = \sum_{B \in E(\Gamma)} \int_0^{|i_B|} p_B(t) dt.$$

Note that in the linear case $W(I)$ is exactly one-half the power dissipated by I (or the energy of I). It turns out that the right mathematical framework in the nonlinear resistive case (2) is the theory of modular sequence spaces, a generalization of Orlicz sequence spaces investigated by Woo [W].

In §§2 and 3 we fix the notation and give a short account of the relevant properties of the modular sequence spaces. In §4 we prove our main result, a Thomson's principle for nonlinear resistive networks. Namely, we show that, under natural assumptions on the external current sources, there is a unique flow I^0 which minimizes W among all flows belonging to the modular sequence space. Such a flow is the unique 1-chain (in such a space) which satisfies Kirchhoff's node and loop laws (in a strengthened form: see Remark 2).

2. NOTATION

Let Γ denote an oriented, infinite, locally finite, connected graph with no multiple edges and no self-loops. We denote by $V(\Gamma)$ the vertex set and by $E(\Gamma)$ the (oriented) edge set. No result presented here will depend on the orientation chosen for the edges of Γ .

Following Flanders [F], a 0-chain

$$I = \sum_{x \in V(\Gamma)} i_x x$$

and a 1-chain

$$I = \sum_{B \in E(\Gamma)} i_B B$$

are defined as formal, real, infinite linear combinations of vertices x or of edges B , respectively.

In the sequel we agree that, whenever $I, J, \dots, Z$ denote 1-chains, $i_B, j_B, \dots, z_B$ will denote the corresponding coefficients of the edge B .

The local finiteness of Γ allows us to define the boundary ∂I of a 1-chain I as the 0-chain

$$\partial I = \sum_{B \in E(\Gamma)} i_B (y(B) - x(B))$$

where $x(B)$ and $y(B)$ are the initial and the terminal point of B , respectively. A 1-chain Z is called a cycle if $\partial Z = 0$.

Next, we fix the assumptions on the functions p_B in (2).

(4a) For every $B \in E(\Gamma)$, p_B is a continuous strictly increasing function on $[0, +\infty)$, and $p_B(0) = 0$.

(4b) There exist a number r and a finite set $\mathcal{F} \subset E(\Gamma)$ such that

$$\frac{tp_B(t)}{M_B(t)} \leq r \quad \text{for all } t \in (0, a_B) \text{ and } B \notin \mathcal{F}$$

where

$$M_B(t) = \int_0^t p_B(u) du$$

and a_B is the unique positive number such that $M_B(a_B) = 1$.

The technical condition (4b) is equivalent to the uniform Δ_2 condition in [W, p. 273] for the functions $N_B(t)$ defined as

$$N_B(t) = M_B(a_B t).$$

Observe that M_B is a strictly convex function and that (4b) implies $M_B(t) \geq (t/a_B)^r$ for $0 \leq t \leq a_B$ and $B \notin \mathcal{F}$ (compare with [W, Proposition 3.2]).

The functional W becomes

$$W(I) = \sum_{B \in E(\Gamma)} M_B(|i_B|).$$

Let p_B^{-1} denote the inverse function of p_B and set

$$M_B^*(t) = \int_0^t p_B^{-1}(u) du.$$

Then M_B and M_B^* are complementary Orlicz functions and the following simple relations hold for all $a, b > 0$ [K-R, pp. 12-13].

$$(5) \quad ab \leq M_B(a) + M_B^*(b) \quad (\text{Young's inequality})$$

$$(6) \quad ap_B(a) = M_B(a) + M_B^*(p_B(a)).$$

3. MODULAR SEQUENCE SPACES AND KIRCHHOFF'S LAWS

Our reference for modular sequence spaces is the above quoted paper by Woo and the book by Lindenstrauss and Tzafriri [L-T]. The standard results in functional analysis used below can be found in Dunford and Schwartz [D-S].

Let the countable set of functions $\{p_B\}$ satisfy assumptions (4a) and (4b) of §2. The modular sequence space $\ell\{M_B\}$ is defined as the linear space of all 1-chains I such that, for some $t > 0$

$$(7) \quad \sum_{B \in E(\Gamma)} M_B \left(\frac{|i_B|}{t} \right) < \infty.$$

For all $I \in \ell\{M_B\}$ set

$$\|I\| = \inf \left\{ t: \sum_{B \in E(\Gamma)} M_B \left(\frac{|i_B|}{t} \right) \leq 1 \right\}.$$

With this norm $\ell\{M_B\}$ becomes a Banach space. We denote by $c\{M_B\}$ the closed linear subspace of all 1-chains I such that (7) holds for every $t > 0$. Since, by (4b), the uniform Δ_2 condition is satisfied by the normalized sequence $\{N_B\}_{B \in E(\Gamma)}$, $\ell\{N_B\} = c\{N_B\}$ and the elementary 1-chains $\{B\}_{B \in E(\Gamma)}$ form an unconditional basis [W, Theorems 3.5 and 4.4].

Woo's results on modular sequence spaces are stated in terms of the normalized spaces $\ell\{N_B\}$ for technical reasons. However, it is easy to prove (see the proof of the Proposition below), that $\ell\{M_B\}$ is isometrically isomorphic to $\ell\{N_B\}$, so that they have the same good topological properties. In particular, if we denote, as usual, by X^* the dual of a Banach space X and let $\cong$ denote isometric isomorphism, we have the following results.

Proposition.

$$(8a) \quad \ell\{M_B\} = c\{M_B\},$$

$$(8b) \quad c\{M_B^*\}^* \cong \ell\{M_B\},$$

$$(8c) \quad \ell\{M_B\}^* \cong \ell\{M_B^*\},$$

and the duality is expressed by the pairing

$$(9) \quad \langle I, J \rangle = \sum_{B \in E(\Gamma)} i_B j_B \quad I \in \ell\{M_B\}, \quad J \in \ell\{M_B^*\}.$$

Proof. Let T denote the following linear transformation on the linear space of all 1-chains

$$T \left(\sum_{B \in E(\Gamma)} i_B B \right) = \sum_{B \in E(\Gamma)} i_B a_B^{-1} B$$

and let $T_{\ell\{M_B\}}$ denote the restriction to $\ell\{M_B\}$. Then, $T_{\ell\{M_B\}}$ is an isometric isomorphism of $\ell\{M_B\}$ onto $\ell\{N_B\}$, mapping $c\{M_B\}$ onto $c\{N_B\}$. Hence (8a) holds (see above).

Now observe that the proof of Proposition 4.1 and of the first statement in Theorem 4.2 in [W] depend neither on normalization nor on uniform Δ_2 condition. Therefore (8b) follows from these results applied to the complementary functions M_B^* and M_B . The same argument and (8a) give (8c) and (9).

Remark 1. Note that (8a) implies that the $W(I)$ is finite for every $I \in \ell\{M_B\}$.

The pairing (9) induces the following equivalent norm on $\ell\{M_B\}$

$$\|I\| = \sup \left\{ \sum_{B \in E(\Gamma)} |i_B j_B| : \sum_{B \in E(\Gamma)} M_B^*(J) \leq 1 \right\}$$

(see [W, Proposition 4.1]).

Lemma 1. Let $I = \sum_{B \in E(\Gamma)} i_B B$ belong to $\ell\{M_B\}$. Then the 1-chain $H = \sum_{B \in E(\Gamma)} p_B(|i_B|)B$ belongs to $\ell\{M_B^*\}$.

Proof. By (6), (4b) and the definition of a_B , we have that, for all $B \notin \mathcal{F}$, $|i_B| < a_B$ implies

$$(11) \quad M_B^*(p_B(|i_B|)) \leq |i_B| p_B(|i_B|) \leq r M_B(|i_B|).$$

Since by (8a)

$$(12) \quad \sum_{B \in E(\Gamma)} M_B(|i_B|) < +\infty,$$

by (4a) and the definition of a_B again, for all but a finite number of edges B , $|i_B| < a_B$. The thesis follows from (11).

Definition. Taking the above Proposition and Lemma 1 into account, we define the resistance operator R from $\ell\{M_B\}$ to $\ell\{M_B^*\}$ as

$$R(I) = \sum_{B \in E(\Gamma)} \text{sign}(i_B) p_B(|i_B|) B, \quad I \in \ell\{M_B\}.$$

In the following we will denote by $\mathcal{Z}$ the linear subspace of all cycles belonging to $\ell\{M_B\}$.

Lemma 2. $\mathcal{Z}$ is a closed subspace of $\ell\{M_B\}$.

Proof. For all $x \in V(\Gamma)$, let E_x denote the finite subset of edges having x as an extreme. For $B \in E_x$ define $\phi_B(x)$ to be 1 or -1 according to $B = [y, x]$ or $B = [x, y]$. One checks immediately that for every 1-chain Z (cf. (9))

$$(13) \quad \partial Z = \sum_{x \in V(\Gamma)} \left(\sum_{B \in E_x} \phi_B(x) z_B \right) x = \sum_{x \in V(\Gamma)} \langle \Phi(x), Z \rangle x$$

where $\Phi(x) = \sum_{B \in E_x} \phi_B(x) B \in \ell\{M_B^*\}$ for all x . The thesis follows by (13) and by Lemma 1.

Let $\iota = \sum_{x \in V(\Gamma)} \iota_x x$ be a 0-chain with the property that there exists $L \in \ell\{M_B\}$ such that $\partial L = \iota$. For instance this is certainly the case if ι is a finite chain and $\sum_{x \in V(\Gamma)} \iota_x = 0$ (see e.g [F, p. 328]).

Definition. We say that a 1-chain $I \in \ell\{M_B\}$ satisfies Kirchhoff's laws if

$$(14) \quad \partial I + i \equiv 0 \quad (\text{node law})$$

$$(15) \quad \langle Z, R(I) \rangle = 0 \quad \text{for all } Z \in \mathcal{Z} \quad (\text{loop law}).$$

In particular, a 1-chain in $\ell\{M_B\}$ satisfying (14) is called a flow. A flow satisfying also (15) is called a current generated by ι .

Remark 2. Equation (15) expresses the fact that the voltage drop is zero along all cycles in $\ell\{M_B\}$, finite or infinite. This strengthened form of Kirchhoff's loop law was formulated by Zemanian [Z1] in Hilbert spaces, and appears to be better adapted to infinite networks. In general this formulation is strictly stronger than that of requiring zero voltage drop only along finite cycles.

4. THOMSON'S PRINCIPLE

In this section we prove our main result announced in §1.

Theorem. Let Γ and the functions p_B be as above. Suppose that $\iota = \partial L$ for some $L \in \ell\{M_B\}$. Then, there exists a unique current $I^0 \in \ell\{M_B\}$ generated by ι , and I^0 is the unique minimum point of W over all flows in $\ell\{M_B\}$.

Proof. Let

$$\mathcal{X} = -L + \mathcal{Z}, \quad \mathcal{X}_n = \{I \in \mathcal{X} : \|I\| \leq n\}.$$

Note that $\mathcal{X}$ is nonempty and $I \in \mathcal{X}$ if and only if I is a flow. For the reader's convenience we split the proof in several steps.

1. W is a strictly convex functional on $\ell\{M_B\}$. In fact, for all B , all 1-chains I, J in $\ell\{M_B\}$, and every $\lambda \in (0, 1)$ we have

$$\begin{aligned} M_B(|\lambda i_B + (1 - \lambda)j_B|) &\leq M_B(\lambda|i_B| + (1 - \lambda)|j_B|) \\ &< \lambda M_B(|i_B|) + (1 - \lambda)M_B(|j_B|) \end{aligned}$$

whence the assertion.

2. W is weak-star lower sequentially semicontinuous on $\ell\{M_B\}$. First recall that, by (8b), $\ell\{M_B\}$ is the dual space of $c\{M_B^*\}$. Let $\{I^{(k)}\} \subset \ell\{M_B\}$ be a sequence converging to I in the weak-star topology. Then $I^{(k)}$ converges to I coordinatewise, i.e.

$$i_B^{(k)} \rightarrow i_B \quad \text{as } k \rightarrow \infty \quad \text{for all } B \in E(\Gamma).$$

By Fatou's lemma

$$\begin{aligned} \liminf_{k \rightarrow \infty} W(I^{(k)}) &= \liminf_{k \rightarrow \infty} \sum_{B \in E(\Gamma)} M_B(|i_B^{(k)}|) \\ &\geq \sum_{B \in E(\Gamma)} \lim_{k \rightarrow \infty} M_B(|i_B^{(k)}|) \\ &= W(I). \end{aligned}$$

3. W is everywhere Gateaux differentiable with G -differential

$$(16) \quad W'(I)Z = \sum_{B \in E(\Gamma)} \text{sign}(i_B) p_B(|i_B|) z_B$$

for all $Z \in \ell\{M_B\}$.

We have to prove that

$$\lim_{t \rightarrow 0} \frac{W(I + tZ) - W(I)}{t} = \sum_{B \in E(\Gamma)} \text{sign}(i_B) p_B(|i_B|) z_B.$$

Now, for every B ,

$$\lim_{t \rightarrow 0} \frac{M_B(|i_B + tz_B|) - M_B(|i_B|)}{t} = \text{sign}(i_B) p_B(|i_B|) z_B.$$

Moreover, for $|t| \leq 1$

$$\left| \frac{M_B(|i_B + tz_B|) - M_B(|i_B|)}{t} \right| \leq |z_B| p_B(|i_B| + |z_B|),$$

and $\sum_{B \in E(\Gamma)} |z_B| p_B(|i_B| + |z_B|) < \infty$ by Lemma 1, (8c), and (9). Therefore (16) follows from Lebesgue dominated convergence theorem.

4. There exists a unique $I^0 \in \mathcal{X}$ which minimizes W on $\mathcal{X}$.

By the definition of $\|\cdot\|$ and (5)

$$\begin{aligned} \|I\| &= \sup \left\{ \sum_{B \in E(\Gamma)} |i_B j_B| : \sum_{B \in E(\Gamma)} M_B^*(|j_B|) \leq 1 \right\} \\ &\leq 1 + W(I), \end{aligned}$$

so that $W(I) \rightarrow +\infty$ as $\|I\| \rightarrow +\infty$.

Hence, there exists n such that

$$\alpha = \inf\{W(I) : I \in \mathcal{X}\} = \inf\{W(I) : I \in \mathcal{X}_n\}.$$

Clearly $\mathcal{X}_n$ is compact in the weak-star topology. As noted in [W, p. 273], $c\{M_B^*\}$ is separable, so that the weak-star topology in $\mathcal{X}_n$ is also metrizable. By Step 2 there exists $I^0 \in \mathcal{X}_n$ such that $\alpha = W(I^0)$ and, by Step 1, I^0 is the unique minimum point of W on $\mathcal{X}$ with this property.

5. I^0 is the unique current generated by ι . Since I^0 minimizes W on $\mathcal{X}$, then

$$(17) \quad W'(I^0)Z = 0 \quad \text{for all } Z \in \mathcal{X}$$

so that I^0 satisfies Kirchhoff's loop law by (16). Since I^0 is the unique minimum point of W on $\mathcal{X}$, by [V, Theorem 9.1], applied to the linear manifold $\mathcal{X}$, I^0 is also the unique element in $\mathcal{X}$ satisfying (17).

This concludes the proof.

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