A BRIEF PROOF OF JACOBIAN HYPOTHESIS IMPLIES FLATNESS

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ABSTRACT. Let S be a polynomial ring in n variables over a field, and let R be the subring generated by n polynomials. We give a short proof of the fact that if the Jacobian determinant of these n polynomials is 1, then S is a flat R -module.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k and let $f_1, f_2, \dots, f_n \in k[x_1, \dots, x_n]$ satisfy the Jacobian hypothesis

$$(*) \quad \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = 1.$$

Then $f_1, f_2, \dots, f_n$ are algebraically independent over k so that $R = k[f_1, \dots, f_n]$ is also a polynomial ring over k . If k has characteristic 0, the Jacobian conjecture asserts that $S = R$, or equivalently, S is a projective R -module [5, Theorem 46 (iii), pp. 483–484]. The Jacobian hypothesis $(*)$ implies that S is smooth over R [4, Theorem, p. 484] and separable over R [5, Theorem 7, p. 458]; moreover, S is finitely generated, equidimensional over R , R is regular, and S is Cohen–Macaulay, hence by Grothendieck's results whose proofs are not easy [1, IV, §15; 2, IV, §17], S is flat over R [5, Theorem 38, (1), (4), p. 477]. In this short note we shall give a brief proof of this fact. Consequences of flatness include that the projective dimension of S over R is at most one and that $S \cap k(f_1, \dots, f_n) = R$ [5, pp. 479–481].

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Theorem. *If the Jacobian determinant*

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = 1,$$

then $S = k[x_1, \dots, x_n]$ *is flat over* $R = k[f_1, \dots, f_n]$.

Proof. In order to show that S is flat over R , we may assume that k is algebraically closed. To see this let $\bar{k}$ denote the algebraic closure of k , then we have by the theorem that $\bar{k}[x_1, \dots, x_n]$ is flat over $\bar{k}[f_1, \dots, f_n]$ and consequently over $k[f_1, \dots, f_n]$. As $\bar{k}$ is free over k , $\bar{k}[x_1, \dots, x_n]$ is faithfully flat over $k[x_1, \dots, x_n]$ and we have that $k[x_1, \dots, x_n]$ is flat over $k[f_1, \dots, f_n]$.

To show S is flat over R it suffices to show that for all maximal ideals $\mathfrak{q}$ of S , the localization $S_{\mathfrak{q}}$ is flat over $R_{\mathfrak{p}}$ where $\mathfrak{p} = \mathfrak{q} \cap R$ [3, Theorem 7.1, p. 46]. A maximal ideal $\mathfrak{q}$ of S looks like $(x_1 - \lambda_1, \dots, x_n - \lambda_n)$ for some $\lambda_1, \dots, \lambda_n \in k$, and $\mathfrak{p} = \mathfrak{q} \cap R = (f_1(x_1, \dots, x_n) - f_1(\lambda_1, \dots, \lambda_n), \dots, f_n(x_1, \dots, x_n) - f_n(\lambda_1, \dots, \lambda_n))$ is a maximal ideal of R . To simplify the notation, let $A = R_{\mathfrak{p}}$, $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ its maximal ideal; similarly, let $B = S_{\mathfrak{q}}$, $\mathfrak{n} = \mathfrak{q}S_{\mathfrak{q}}$ its maximal ideal.

It is well known that $(B, \mathfrak{n}, k)$ is an n -dimensional regular local ring so that its maximal ideal $\mathfrak{n}$ can be generated by n elements, any such n elements are called a *regular system of parameters* of B . Moreover, if $(v_1, \dots, v_n)B = \mathfrak{n}$, then there is an isomorphism of graded rings:

$$\begin{aligned} k[t_1, \dots, t_n] &\rightarrow \text{gr}_{\mathfrak{n}}(B) \\ t_i &\mapsto v_i + \mathfrak{n}^2 \end{aligned}$$

for all i . Here $k[t_1, \dots, t_n]$ is a polynomial ring over k and

$$\text{gr}_{\mathfrak{n}}(B) = \bigoplus_{i=0}^{\infty} \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}$$

is the graded ring of B associated with powers of $\mathfrak{n}$.

Similarly, $(A, \mathfrak{m}, k)$ is an n -dimensional regular local ring so that if $\mathfrak{m} = (u_1, \dots, u_n)A$, then there is an isomorphism of graded rings: $k[t_1, \dots, t_n] \rightarrow \text{gr}_{\mathfrak{m}}(A)$ given by $t_i \mapsto u_i + \mathfrak{m}^2$ for all i .

In what follows, let

$$u_i = \frac{f_i(x_1, \dots, x_n) - f_i(\lambda_1, \dots, \lambda_n)}{1}$$

which is an element of A . Clearly, $(u_1, \dots, u_n)A = \mathfrak{m}$. Next we want to show that the Jacobian hypothesis $(*)$ implies that $(u_1, \dots, u_n)B = \mathfrak{n}$, i.e., a (hence any) regular system of parameters of A is also a regular system of parameters of B , i.e.,

$$(**) \quad \mathfrak{m}B = \mathfrak{n}.$$

Using Taylor expansion, we have the following matrix equation:

$$\begin{bmatrix} f_1(x_1, \dots, x_n) - f_1(\lambda_1, \dots, \lambda_n) \\ \vdots \\ f_n(x_1, \dots, x_n) - f_n(\lambda_1, \dots, \lambda_n) \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{\lambda}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{\lambda}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{\lambda}) & \dots & \frac{\partial f_n}{\partial x_n}(\vec{\lambda}) \end{bmatrix} \begin{bmatrix} x_1 - \lambda_1 \\ \vdots \\ x_n - \lambda_n \end{bmatrix} \pmod{q^2}.$$

Here $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$. Passing to the localization S_q , we have

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{\lambda}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{\lambda}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{\lambda}) & \dots & \frac{\partial f_n}{\partial x_n}(\vec{\lambda}) \end{bmatrix} \begin{bmatrix} \frac{x_1 - \lambda_1}{1} \\ \vdots \\ \frac{x_n - \lambda_n}{1} \end{bmatrix} \pmod{n^2}.$$

Since the Jacobian matrix is invertible by (*) the last equation shows that the images of $u_1, \dots, u_n$ form a basis of the vector space n/n^2 over k . Hence by a version of NAK (Nakayama's lemma) [3, Theorem 2.3 (i), p. 8], $u_1, \dots, u_n$ generate n . This establishes (**).

As a result of (**), we have the following commutative diagram

$$\begin{array}{ccc} k[t_1, \dots, t_n] & \xrightarrow[\sim]{\alpha} & \text{gr}_m(A) \\ \parallel & & \downarrow \gamma \\ k[t_1, \dots, t_n] & \xrightarrow[\sim]{\beta} & \text{gr}_n(B) \end{array}$$

where $\alpha: t_i \mapsto u_i + m^2$ and $\beta: t_i \mapsto u_i + n^2$ are isomorphisms, and γ is induced by the inclusion $A \hookrightarrow B$. Thus γ is an isomorphism, and B is flat over A by the local flatness criterion [4, Theorem 22.3, (1), (4'), p. 174]. $\square$

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