PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 109, Number 3, July 1990

A PROPERTY OF PURELY INFINITE SIMPLE C^*-ALGEBRAS

SHUANG ZHANG

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. An alternative proof is given for the fact ([13]) that a purely infinite, simple C^*-algebra has the FS property: the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. In particular, the Cuntz algebras O_n $(2 \leq n \leq +\infty)$ and the Cuntz–Krieger algebras O_A, if A is an irreducible matrix, have the FS property. This answers a question raised in [2, 2.10] concerning the structure of projections in the Cuntz algebras. Moreover, many corona algebras and multiplier algebras have the FS property.

A C^*-algebra A is said to be purely infinite if $(xAx)^-$ contains an infinite projection for every nonzero positive element x in A ([7, 12]). The author recently proved ([13]) that purely infinite, simple C^*-algebras have the FS property; namely, the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. Actually, many interesting C^*-algebras have the FS property. For example, the Bunce–Deddens algebras have FS ([1, 3]); many corona algebras and multiplier algebras have FS ([5, 13]); certain irrational rotation algebras have FS ([6]). Certainly, all AF algebras, von Neumann algebras, and AW^* algebras have FS.

In this short note, we provide another proof for the fact that purely infinite, simple C^*-algebras have the FS property. The algebras O_n $(2 \leq n \leq +\infty)$ and O_A, if A is an irreducible matrix, are purely infinite and simple ([7, 8, 9]), and many corona algebras are purely infinite and simple ([12, 13]). Hence, these C^*-algebras have the FS property. In particular, this answers a question of B. Blackadar raised in [2, 2.10] concerning the projection structure of the Cuntz algebras.

1. **Theorem.** *If A is a purely infinite, simple C^*-algebra, then A has the FS property, and hence $RR(A) = 0$.*

Proof. To prove the conclusion, by [2, 2.7; 10], it is equivalent to prove that every hereditary C^*-subalgebra of A has an approximate identity consisting of

Received by the editors April 17, 1989 and, in revised form, September 5, 1989; the results in this paper were presented at the 17th Annual Canadian Symposium on Operator Algebras and Operator Theory, University of Toronto, May 22–26, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L05.

Key words and phrases. Purely infinite simple C^*-algebras, projections, the Cuntz algebras.

This research was partially supported by a grant from the National Science Foundation.

projections (the HP property). It suffices to show that for any positive element x in A and any positive number δ, there exists a projection p in the hereditary C^*-subalgebra A_x of A generated by x such that

$$\|(1-p)x\| < \delta .$$

We can assume that $\|x\| = 1$. If $0 \notin \sigma(x)$, then $A_x = A$ and A has a unit. Let p be the unit. If 0 is an isolated point of $\sigma(x)$, then let p be the spectral projection $E_{(0,\infty)}(x)$ of x over the interval $(0, \infty)$, which is in A_x. Hence, we can, furthermore, assume that 0 is a limit point of $\sigma(x)$. From now on, we will denote the Banach space double dual of A by A^{**}.

For any positive number ε, we define a real-valued continuous function $f_\varepsilon(t)$ on the interval $[0, 1]$ as follows:

$$f_\varepsilon(t) = \begin{cases} t & \text{if } 2\varepsilon \le t , \\ \text{linear} & \text{if } \varepsilon \le t \le 2\varepsilon , \\ 0 & \text{if } t \le \varepsilon . \end{cases}$$

Clearly,

$$\|x - f_\varepsilon(x)\| < 2\varepsilon .$$

Let B_ε be the hereditary C^*-subalgebra of A supported by the open projection $p_\varepsilon = E_{(\varepsilon,\infty)}(x)$, where $E_{(\varepsilon,\infty)}(x)$ is the spectral projection of x in A^{**} over the interval (ε, ∞). Clearly, $f_\varepsilon(x)$ is a strictly positive element of B_ε, and hence B_ε is σ-unital. Similarly, let A_ε to be the hereditary C^*-subalgebra of A supported by the open projection $E_{(0,\varepsilon)}(x)$. It is obvious that A_ε and B_ε are mutually orthogonal, purely infinite, and simple.

Since 0 is a limit point of $\sigma(x)$, we can choose a nonzero projection r in A_ε. Since A is purely infinite and simple, by a routine argument (for example, see the proof of [2, 3.12]), we can obtain a sequence of nonzero subprojections of r, say $\{q_i\}$, such that

$$q_i \sim q_j \quad \text{if } i, j \ge 1, \quad \text{and} \quad q_j q_i = 0 \quad \text{if } i \ne j ,$$

where '$\sim$' means the Murray–von Neumann equivalence of projections in A.

Set $p_0 = \sum_{i=1}^\infty q_i$. It is easily verified that p_0 is an open projection in A^{**} and the hereditary C^*-subalgebra B_0 of A supported by p_0 is simple and stable; actually, $B_0 \cong q_1 A q_1 \otimes K$, where K is the C^*-algebra consisting of all compact operators on a separable Hilbert space. Since both $q_1 A q_1$ and B_ε are σ-unital full hereditary C^*-subalgebras of A_x and A_x is σ-unital also, by [4, 2.8], we have that $B_0 \cong B_\varepsilon \otimes K$. Obviously, B_0 is a hereditary C^*-subalgebra of A_ε, and of course is orthogonal to B_ε. Let $q_0 = p_\varepsilon + p_0$. Then q_0 is an open projection in A^{**} and the hereditary C^*-subalgebra B_1 of A_x supported by q_0 is *-isomorphic to $B_\varepsilon \otimes K$. Hence B_1 is *-isomorphic to $q_1 A q_1 \otimes K$. It follows that B_1 has an approximate identity consisting of projections. Thus, we can find a projection p in $B_1 \subset A_x$ such that

$$\|(1-p) f_\varepsilon(x)\| < \varepsilon .$$

Therefore,

$$\|(1-p)x\| \leq \|(1-p)(x-f_\varepsilon(x))\| + \|(1-p)f_\varepsilon(x)\| < 3\varepsilon .$$

Since ε can be arbitrarily small, this completes the proof. □

2. **Corollaries.** (i). *The Cuntz algebras* O_n $(2 \leq n \leq \infty)$ *and the Cuntz–Krieger algebra* O_A, *if* A *is an irreducible matrix, have the FS property.*

(ii). *If* A *is a* σ*-unital, nonunital, simple* C^**-algebra with the FS property, then* $M(A)/A$ *has the FS property provided* $M(A)/A$ *is simple. If, in addition, every projection in* $M(A)/A$ *lifts to a projection in* $M(A)$, *then* $M(A)$ *has the FS property.*

Proof. (i) follows from Theorem 1 and [7, 1.6]. (ii) follows from Theorem 1, [12, 1.3] and [13] or [5]. □

3. *Remark.* The author has pointed out in [11, 3.1], under the assumption that A is a σ-unital C^*-algebra with the FS property, that $M(A)$ has the FS property if and only if every self-adjoint element of $M(A)$ can be written in the following form:

$$\sum_{i=1}^{\infty} \lambda_i p_i + a ,$$

where $\{\lambda_i\}$ is a bounded real sequence, $\{p_i\}$ is a sequence of mutually orthogonal projections of A, and a is a self-adjoint element of A. In other words, the general Weyl–von Neumann theorem holds in $M(A)$, if and only if $M(A)$ has FS. The reader is referred to [5, 11, 13, 14] for more examples of C^*-algebras with the FS property and related results.

REFERENCES

1. J. Bunce and J. Deddens, *A family of simple C^*-algebras related to weighted shift operators*, J. Funct. Anal. **19** (1975), 13–24.
2. B. Blackadar, *Notes on the structure of projections in simple algebras*, Semesterbericht Funktionalanalysis, Tubingen, Wintersemester, 1982/83.
3. B. Blackadar and A. Kumjian, *Skew products of relations and structure of simple C^*-algebras*, Math. Z. **189** (1985), 55–63.
4. L. G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), 335–348.
5. L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, preprint, 1989.
6. M.-D. Choi and G. Elliott, *Density of the self-adjoint elements with finite spectrum in an irrational rotation C^*-algebra*, preprint.
7. J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **131** (1981), 181–197.
8. ——, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173–185.
9. J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Inventiones Math. **56** (1980), 251–268.
10. G. K. Pedersen, *The linear span of projections in simple C^*-algebras*, J. Operator Theory **4** (1980), 289–296.
11. S. Zhang, *K_1-groups, quasidiagonality and interpolation by multiplier projections*, Trans. Amer. Math. Soc. (to appear).

12. ——, *On the structure of projections and ideals of corona algebras*, Canad. J. Math. **41** (1989), 721–742.

13. ——, *C^*-algebras with real rank zero and the internal structure of their corona and multiplier algebras*, Part I, Part II, Part IV, preprints.

14. ——, *C^*-algebras with real rank zero and the internal structure of their corona and multiplier algebras*, Part III, Canad. J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045