WHEN DOES $ca(\Sigma, X)$ CONTAIN A COPY OF l_∞ OR c_0 ?

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ABSTRACT. For a σ -algebra Σ and a Banach space X , $ca(\Sigma, X)$ is the Banach space of all vector measures from Σ to X . If Σ admits a nonzero atomless finite positive measure, then $ca(\Sigma, X) \supset l_\infty$ (or c_0) if and only if there is a noncompact bounded linear operator from l_2 to X (Theorem 1). Otherwise, $ca(\Sigma, X) \supset l_\infty$ (or c_0) if and only if $X \supset l_\infty$ (or c_0) (Theorem 2).

Let Σ be a σ -algebra on a nonempty set S , and X be a Banach space. Then $ca(\Sigma, X)$ denotes the Banach space of all countably additive vector measures $\mu: \Sigma \rightarrow X$ under the sup-norm $\|\mu\| = \sup_{E \in \Sigma} \|\mu(E)\|$. (This norm is of course equivalent with the more commonly used semivariation norm.) If Z is a Banach space, we write $Z \supset l_\infty$ (or c_0) to indicate that Z contains an isomorphic copy of l_∞ (or c_0).

The purpose of this paper is to give a complete answer, in terms of Σ and X , to the question stated in the title. In [5, Corollary 4] it has been recently shown that the subspace $cca(\Sigma, X)$ of $ca(\Sigma, X)$, consisting of measures with relatively norm compact ranges, contains an isomorph of l_∞ if and only if $X \supset l_\infty$. Somewhat surprisingly, it is in general not true that $ca(\Sigma, X) \supset l_\infty$ implies $X \supset l_\infty$ (the converse is always valid trivially). Here are our main results.

Theorem 1. *Suppose the σ -algebra Σ admits a nonzero atomless finite positive measure. Then the following are equivalent.*

- (a) $ca(\Sigma, X) \supset l_\infty$.
- (b) *There exists a noncompact bounded linear operator from l_2 to X .*
- (c) $ca(\Sigma, X) \supset c_0$.

Theorem 2. *Suppose every nonzero finite positive measure on Σ is purely atomic. Then:*

- (A) $ca(\Sigma, X) \supset l_\infty \Leftrightarrow X \supset l_\infty$.
- (B) $ca(\Sigma, X) \supset c_0 \Leftrightarrow X \supset c_0$.

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Of course, the assumptions on Σ in these two theorems are mutually exclusive. Two standard examples of such distinct type Σ s are of course $\Sigma = \mathcal{B}([0, 1])$, the Borel σ -algebra on $[0, 1]$, and $\Sigma = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N} = \{1, 2, \dots\}$.

As an illustration to Theorem 1, let's take $\mathcal{B} = \mathcal{B}([0, 1])$ for Σ . Then $ca(\mathcal{B}, c_0)$ and $ca(\mathcal{B}, l_p)$ for $2 \leq p < \infty$ all contain isomorphic copies of l_∞ . On the other hand, using both Theorems 1 and 2 together with the fact that every operator $l_2 \rightarrow l_p$ ($1 \leq p < 2$) is compact [7, Proposition 2.c.3], we see that for any σ -algebra Σ , $ca(\Sigma, l_p) \not\subset l_\infty$ if $1 \leq p < 2$. (For $p = 1$ this also follows from the result about $cca(\Sigma, X)$ mentioned above because $ca(\Sigma, l_1) = cca(\Sigma, l_1)$ by the Schur property of l_1 .)

Condition (b) in Theorem 1 has an equivalent reformulation:

(b') *There exists a sequence $(x_n) \in l_w^2(X)$ that is not norm null.*

(Clearly, if such a sequence exists, then it can be chosen so that $\|x_n\| = 1$ for all n .) Here, $l_w^2(X) = l_{\text{weak}}^2(X)$ (see [2, Chapter VI]) is the space of all sequences (x_n) in X for which

$$\sum_{n=1}^{\infty} |x^*(x_n)|^2 < \infty, \quad \forall x^* \in X^*.$$

The following, easily verified result can be found in [1, Proposition B]. Here and in what follows we denote by (e_n) the sequence of standard unit vectors in l_∞ and other sequence spaces.

Lemma 1. *Let (x_n) be a sequence in X . Then the following are equivalent.*

- (α) $(x_n) \in l_w^2(X)$.
- (β) *The series $\sum_{n=1}^{\infty} a_n x_n$ converges (unconditionally) in X for all $(a_n) \in l_2$.*
- (γ) *There exists a bounded linear operator $T: l_2 \rightarrow X$ such that $Te_n = x_n$ for all n . (Of course, such a T must be given by the formula $Ta = \sum_{n=1}^{\infty} a_n x_n$ for all $a = (a_n) \in l_2$.)*

Let's now verify that conditions (b) and (b') are indeed equivalent.

(b) $\Rightarrow$ (b'): Let $T: l_2 \rightarrow X$ be a noncompact operator. Then, as easily seen, there exists a sequence (u_n) in l_2 such that $\|u_n\| = 1$, $u_n \rightarrow 0$ weakly, and $\|Tu_n\| > \varepsilon > 0$ for all n . By passing to a subsequence if necessary, we may also assume that (u_n) is in $l_w^2(X)$ (see [1; Proposition D]), and then it is clear that $(x_n) \equiv (Tu_n)$ is as required in (b').

(b') $\Rightarrow$ (b): Let (x_n) be a member of $l_w^2(X)$ with $\|x_n\| = 1$ for all n . Then the operator $T: l_2 \rightarrow X$ defined as in Lemma 1(c) above is obviously noncompact.

The proof of Theorems 1 and 2 will be based on the following three lemmas. We refer the reader to [2], [3], and [6] for the fundamental results on scalar

and vector measures (such as the Orlicz-Pettis theorem, the Vitali-Hahn-Saks theorem, etc.) of which we make free use below. The lemma below, and its proof, were inspired by [1, Theorem 5].

Lemma 2. *Let (μ_n) be a sequence in $ca(\Sigma, X)$ such that the series*

$$\sum_{n=1}^{\infty} \mu_n(E)$$

converges unconditionally in X for all $E \in \Sigma$. Then

$$(\mu(A_n)) \in l_w^2(X)$$

for every sequence (A_n) in Σ .

Proof. Let $x^* \in X^*$. Then, for each $E \in \Sigma$, the series

$$\sum_{n=1}^{\infty} x^* \mu_n(E)$$

converges unconditionally. Hence, by the weak convergence criterion in $ca(\Sigma)$, the series $\sum_{n=1}^{\infty} x^* \mu_n$ is weakly subseries convergent in $ca(\Sigma)$. By the Orlicz-Pettis theorem, it is norm unconditionally convergent in $ca(\Sigma)$. Now, by a theorem about unconditionally convergent series in $L_1(0, 1)$, due to Orlicz (see [2, p. 108]), which extends easily to $ca(\Sigma)$, it follows that

$$\sum_{n=1}^{\infty} \|x^* \mu_n\|^2 < \infty.$$

Finally, if (A_n) is any sequence in Σ , then $|x^* \mu_n(A_n)| \leq \|x^* \mu_n\|$ for all n , and so

$$\sum_{n=1}^{\infty} |x^* \mu_n(A_n)|^2 < \infty. \quad \square$$

The next result is an analog for vector measures of the Phillips lemma (see [2, p. 83]).

Lemma 3. *Let (μ_n) be a sequence in $ca(\mathcal{P}(\mathbf{N}), X)$ such that*

$$\mu_n(E) \rightarrow 0, \quad \forall E \subset \mathbf{N}.$$

Then

$$\|\mu_n\| \rightarrow 0.$$

Proof. If not, then we may assume that for some $\varepsilon > 0$,

$$\|\mu_n\| > 1 + \varepsilon, \quad \forall n \in \mathbf{N}.$$

Since $\|\mu_1\| > 1$, there is $A_1 \subset \mathbf{N}$ for which $\|\mu_1(A_1)\| > 1$. Since

$$\|\mu_1(A_1 \cap \{m, m+1, \dots\})\| \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

there exists m_1 such that if $B_1 = A_1 \cap \{1, \dots, m_1\}$, then

$$\|\mu_1(B_1)\| > 1.$$

Let $n_1 = 1$.

In the second step, since $\mathcal{P}(\{1, \dots, m_1\})$ is finite and $\mu_n \rightarrow 0$ setwise, we find $n_2 > n_1$ such that $\|\mu_{n_2}(E)\| < \varepsilon$ for all $E \subset \{1, \dots, m_1\}$. Since $\|\mu_{n_2}\| > 1 + \varepsilon$, there is $A_2 \subset \mathbf{N}$ such that $\|\mu_{n_2}(A_2)\| > 1 + \varepsilon$. Then

$$\|\mu_{n_2}(A_2 \cap \{m_1 + 1, m_1 + 2, \dots\})\| > 1,$$

and we can find $m_2 > m_1$ so that if $B_2 = A_2 \cap \{m_1 + 1, \dots, m_2\}$, then

$$\|\mu_{n_2}(B_2)\| > 1.$$

Continuing in this fashion, we find $n_1 < n_2 < \dots$ and a sequence (B_k) of (finite) disjoint subsets of $\mathbf{N}$ such that

$$\|\mu_{n_k}(B_k)\| > 1, \quad \forall k,$$

thus contradicting Nikodym's uniform countable additivity theorem. $\square$

Remark 1. Here we indicate an alternative proof of the above lemma: Let v be the positive measure on $\mathcal{P}(\mathbf{N})$ defined by $v(A) = \sum_{k \in A} 2^{-k}$, and let d be the associated Fréchet-Nikodym metric on $\mathcal{P}(\mathbf{N})$: $d(A, B) = v((A \setminus B) \cup (B \setminus A))$. Then $(\mathcal{P}(\mathbf{N}), d)$ is a compact metric space, homeomorphic to $\{0, 1\}^{\mathbf{N}}$. Let (μ_n) be as in the lemma. By the Vitali-Hahn-Saks theorem, the sequence (μ_n) is equi- v -absolutely continuous, i.e., equicontinuous on $(\mathcal{P}(\mathbf{N}), d)$. Hence, as $\mu_n \rightarrow 0$ pointwise, it follows that $\mu_n \rightarrow 0$ uniformly on $\mathcal{P}(\mathbf{N})$.

Lemma 4. (α) If $ca(\Sigma, X) \supset l_\infty$ while $X \not\supset l_\infty$, or if $ca(\Sigma, X) \supset c_0$ while $X \not\supset c_0$, then there is a sequence (μ_n) in $ca(\Sigma, X)$ such that $\inf_n \|\mu_n\| > 0$ and (*) the series $\sum_{n=1}^\infty \mu_n(E)$ converges unconditionally in X for all $E \in \Sigma$.

(β) Conversely, if a sequence (μ_n) satisfying (*) and such that $\|\mu_n\| \not\rightarrow 0$ exists in $ca(\Sigma, X)$, then $ca(\Sigma, X) \supset l_\infty$.

Proof. (α) Let J be an isomorphic embedding of l_∞ (resp., c_0) into $ca(\Sigma, X)$ and denote $\mu_n = J e_n (n \in \mathbf{N})$. Clearly, $\inf_n \|\mu_n\| > 0$.

Case 1. $ca(\Sigma, X) \supset l_\infty$, but $X \not\supset l_\infty$. Since $X \not\supset l_\infty$, using a result due to Rosenthal (see [8] or [3, p. 149]) we see that for each E in Σ the operator

$$J_E: l_\infty \rightarrow X; a \rightarrow (Ja)(E)$$

is weakly compact. Hence, as is well known (see [3, p. 148] or [5, Lemma 1]), it follows that the series $\sum_{n=1}^\infty J_E e_n \equiv \sum_{n=1}^\infty \mu_n(E)$ converges unconditionally.

Case 2. $ca(\Sigma, X) \supset c_0$ but $X \not\supset c_0$. Since (e_n) is a Schauder basis for c_0 , $Ja = \sum_{n=1}^\infty a_n \mu_n$ (norm convergence) for all $a = (a_n)$ in c_0 . Let $E \in \Sigma$. Then the series $\sum_{n=1}^\infty a_n \mu_n(E)$ converges in X for all zero sequences (a_n) ; that is, the series $\sum_{n=1}^\infty \mu_n(E)$ is weakly unconditionally Cauchy [2, p. 44]. Now, as $X \not\supset c_0$, we can appeal to a result of Bessaga and Pełczyński (see [2, p. 45] or [3, p. 22]) and get (*).

(β) For every $a = (a_n) \in l_\infty$ and $E \in \Sigma$, let

$$(Ta)(E) = \sum_{n=1}^{\infty} a_n \mu_n(E);$$

the series on the right converges by the bounded multiplier property of unconditionally convergent series. Moreover, by the Nikodym countable additivity theorem, the set function $(Ta)(\cdot)$ thus defined is in $ca(\Sigma, X)$. Finally, an easy application of the Banach-Steinhaus or closed graph theorem shows that the linear operator $T: l_\infty \rightarrow ca(\Sigma, X)$ thus obtained is continuous. Now observe that $\|Te_n\| = \|\mu_n\| \not\rightarrow 0$; hence, by an l_∞ theorem of Rosenthal (see [8] or [4]), there exists an infinite subset M of $\mathbb{N}$ such that $J = T|_{l_\infty(M)}$ is an isomorphic embedding; here $l_\infty(M) = \{a \in l_\infty: a_n = 0 \text{ for } n \notin M\} \cong l_\infty$. $\square$

Proof of Theorem 1. (a) $\Rightarrow$ (b): This is obvious when $X \supset l_\infty$, so let us assume that $X \not\supset l_\infty$. Then we can apply Lemma 4(α) and find a sequence (μ_n) satisfying its conclusion. Since $\inf_n \|\mu_n\| > 0$, there is a sequence (A_n) in Σ such that denoting $x_n = \mu_n(A_n)$ we have $\inf_n \|x_n\| > 0$. On the other hand, Lemma 2 tells us that $(x_n) \in l_w^2(X)$. We have thus showed condition (b') which, as we know, is equivalent to (b).

(b') $\Rightarrow$ (a): (Much of our inspiration here comes from [1], Theorem 3, and its proof.) By our assumption on Σ , there exists an atomless probability measure v on Σ . Hence we can construct a Rademacherlike sequence (r_n) of real-valued functions on S ; that is, an orthonormal sequence (r_n) in $L_2(v)$ such that $v(r_n = 1) = v(r_n = -1) = 1/2$ for all n . Choose $(x_n) \in l_w^2(X)$ with $\|x_n\| = 1$ for all n . For each n define $\mu_n \in ca(\Sigma, X)$ by

$$\mu_n(E) = 2 \int_E r_n dv \cdot x_n.$$

It is easy to calculate that $\|\mu_n\| = 1$. Moreover, by Parseval's inequality we have $\sum_{n=1}^{\infty} (\int_E r_n dv)^2 \leq v(E) < \infty$ for every $E \in \Sigma$ and, appealing to Lemma 1, we see that the series $\sum_{n=1}^{\infty} \mu_n(E)$ converges unconditionally. From Lemma 4(β) we now get that $ca(\Sigma, X) \supset l_\infty$, i.e., condition (a).

(c) $\Rightarrow$ (a): If $X \supset c_0$, then X has trivially property (b) and this, as we have just seen, implies (a). If $X \not\supset c_0$, then we get (a) by first applying part (α) of Lemma 4 and next part (β).

(a) $\Rightarrow$ (c): Trivial. $\square$

Proof of Theorem 2. Of course, only the implications $\Rightarrow$ are to be shown; we will prove both of them simultaneously. Suppose $X \not\supset l_\infty$ (resp., $X \not\supset c_0$), and let (μ_n) be a sequence in $ca(\Sigma, X)$ provided by Lemma 4(α). Applying a theorem due to Bartle, Dunford, and Schwartz (see [3, p. 14]), it is easy to find a finite positive measure v on Σ such that $\mu_n \ll v$ for all n . By our assumption about Σ , v must be purely atomic. Let (A_k) be a partition of S such that each A_k is an atom of v . Since $\mu_n \ll v$, each A_k is an atom or a

null set for μ_n ($n \in \mathbb{N}$). It follows that if we define $\tilde{\mu}_n \in ca(\mathcal{P}(\mathbb{N}), X)$ by

$$\tilde{\mu}_n(B) = \mu_n(\cup_{k \in B} A_k),$$

then the series

$$\sum_{n=1}^{\infty} \tilde{\mu}_n(B)$$

will converge (unconditionally) and hence $\tilde{\mu}_n(B) \rightarrow 0$ for every B in $\mathcal{P}(\mathbb{N})$. By Lemma 3 we should have $\|\tilde{\mu}_n\| \rightarrow 0$, but on the other hand, $\|\tilde{\mu}_n\| = \|\mu_n\| \not\rightarrow 0$: a contradiction. $\square$

Remark 2. For a set Γ and a Banach space X , let $l_1[\Gamma, X]$ denote the Banach space of all summable families $(x_\gamma)_{\gamma \in \Gamma}$ in X , with the norm $\|(x_\gamma)\| = \sup\{\|\sum_{\gamma \in \Delta} x_\gamma\| : \Delta \subset \Gamma\}$. Suppose a σ -algebra Σ admits no atomless probability measure. Then it can be shown that there exists a set Γ such that, for any Banach space X , $ca(\Sigma, X)$ is isometrically isomorphic to $l_1[\Gamma, X]$. Conversely, given Γ , $l_1[\Gamma, X] \cong ca(\Sigma, X)$, where Σ is the σ -algebra of all countable-cocountable subsets of Γ . Therefore, Theorem 2 can be viewed as a reformulation of the following (certainly well-known) result: $l_1[\Gamma, X] \supset l_\infty$ (or c_0) if and only if $X \supset l_\infty$ (or c_0).

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