SPECTRAL PICTURES OF HYPONORMAL BILATERAL OPERATOR WEIGHTED SHIFTS

DOMINGO A. HERRERO

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ABSTRACT. This note provides a complete description of the spectral picture of a hyponormal bilateral operator weighted shift of finite multiplicity. If the weights are $m \times m$ matrices, then the operator cannot be cyclic, unless it is a normal operator of a very special kind. An example shows that, nevertheless, there exists a cyclic, nonnormal, hyponormal bilateral operator weighted shift whose weights are operators acting on an infinite dimensional Hilbert space. (This answers a question of J. B. Conway.)

1. INTRODUCTION

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all (bounded linear) operators acting on a complex, separable Hilbert space $\mathcal{H}$, and let $\{A_n\}(-\infty < n < \infty)$ be a uniformly bounded two-sided sequence of operators. The *bilateral operator weighted shift* B (with weight sequence $\{A_n\}$) is the operator on $\ell^2(\mathcal{H}) := \ell^2 \otimes \mathcal{H}$ defined by

$$B(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0x_0, \dots),$$

where $x = (\dots, x_{-1}, [x_0], x_1, \dots) \in \ell^2(\mathcal{H})$; $[x_0]$ denotes the central (0th) term of the two-sided sequence.

The adjoint of B is defined by a similar equality

$$B^*(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_{-1}^*x_0, [A_0^*x_1], A_1^*x_2, \dots),$$

and the self-commutator $[B^*, B] = B^*B - BB^*$ of B is the “diagonal” hermitian operator given by

$$\begin{aligned} [B^*, B](\dots, x_{-1}, [x_0], x_1, \dots) = & (\dots, A_{-1}^*A_{-1} - A_{-2}, A_{-2}^*)x_{-1}, \\ & [(A_0^*A_0 - A_{-1}A_{-1}^*)x_0], (A_1^*A_1 - A_0A_0^*)x_1, \dots). \end{aligned}$$

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It readily follows that B is *hyponormal* (that is, $[B^*, B] \geq 0$) if and only if

$$A_n^* A_n \geq A_{n-1}^* A_{n-1} \quad \text{for all } n.$$

Straightforward calculations show that $\|B\| = \sup_n \|A_n\|$ and that if A_n admits a polar decomposition $A_n = U_n H_n$ with U_n unitary ($0 \leq H_n \leq \|A_n\|$), then B is unitarily equivalent to the operator defined in the same way, with weights $\{H_n\}$. Unless otherwise stated, we shall assume that this is the case; that is, $0 \leq A_n \leq \|B\|$ for all n . In this case, the hyponormality of B can be expressed as

$$A_n^2 \geq A_{n-1}^2 \quad \text{for all } n,$$

and this implies that $\{A_n\}$ is a bounded, nondecreasing (see [1, 2, 9]) sequence of positive hermitian operators. Therefore, the strong limits

$$\lim_{n \rightarrow \infty} A_{-n} = A_- \quad \text{and} \quad \lim_{n \rightarrow \infty} A_n = A_+$$

are well defined [11, Problem 94]. Clearly, $0 \leq A_- \leq A_n \leq A_+$ for all n ; moreover, since B is hyponormal, the spectral radius $\text{sp}(B)$ of B coincides with $\|B\|$, and either A_- is not invertible and $0 \in \sigma(B)$ or A (and therefore B) is invertible, B^{-1} is a hyponormal and $\text{sp}(B^{-1}) = \|A_-^{-1}\|$ [16, Chapter 3].

Since B is a bilateral operator weighted shift, every piece of $\sigma(B)$ is invariant under rotation [19]; therefore

$$\sigma(B) \supset \{\lambda \in \mathbb{C} : |\lambda| = \|B\| = \|A_+\| \text{ or } |\lambda| = \|A_-^{-1}\|^{-1}\}.$$

($\|A_-^{-1}\|^{-1}$ is defined equal to 0 in case A_- is not invertible).

In what follows, $\sigma_{\ell e}(T)$, $\sigma_{\text{re}}(T)$, $\sigma_e(T)$ and $\sigma_{\ell \text{re}}(T)$ will denote the left essential spectrum of T , the right essential spectrum of T , the essential spectrum of T and, respectively, $\sigma_{\ell e}(T) \cap \sigma_{\text{re}}(T)$. Recall that $\mathbb{C} \setminus \sigma_{\ell \text{re}}(T)$ coincides with $\rho_{\text{s-F}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not a semi-Fredholm operator}\}$. (The reader is referred to [7] for a definition and properties of the semi-Fredholm operators.)

Two operators, A and B , acting on (not necessarily the same) Hilbert spaces are called *approximately unitarily equivalent* if $\|A - U_k B U_k^*\| \rightarrow 0$ ($k \rightarrow \infty$) for a suitable sequence of unitary mappings $\{U_k\}$ from the space of B onto the space of A . (In symbols: $A \simeq_a B$. In this case the U_k 's can be chosen so that, in addition, $A - U_k B U_k^*$ is a compact operator for all $k = 1, 2, \dots$ [20].)

The *reducing essential spectrum* of an operator T is the (possibly empty) compact set

$$\sigma_{\text{red}}(T) = \{\lambda \in \mathbb{C} : T \simeq_a T \oplus \lambda\}$$

($\sigma_{\text{red}}(T) \subset \sigma_{\ell \text{re}}(T)$; this notion was introduced by N. Salinas in [18]).

Let U denote the bilateral shift on ℓ^2 ; then

$$U \simeq \begin{pmatrix} S & Z \\ 0 & S^* \end{pmatrix}_{\ell^2_+ \ell^2_-}$$

where S is the unilateral shift on ℓ^2_+ .

With this notation in mind, we have the following.

Theorem 1. Let B be a bilateral operator weighted shift on $\ell^2(\mathcal{H})$ whose weight sequence $\{A_n\} (0 \leq A_n \leq \|B\|)$ admits strong limits in both directions:

$$\lim_{n \rightarrow \infty} A_{-n} = A_- \quad \text{and} \quad \lim_{n \rightarrow \infty} A_n = A_+.$$

Then B is approximately unitarily equivalent to the direct sum of B and the normal operator $U \otimes (A_- \oplus A_+)$. In particular,

$$\sigma_{\text{red}}(B) \supset \sigma(U \otimes (A_- \oplus A_+)) = \{\lambda \in \mathbb{C} : |\lambda| \in \sigma(A_-) \cup \sigma(A_+)\}.$$

Furthermore, if $1 \leq \dim \mathcal{H} < \infty$, then

$$\sigma_e(B) = \sigma_{\ell \text{ re}}(B) = \sigma_{\text{red}}(B) = \sigma(U \otimes (A_- \oplus A_+)).$$

Theorem 2. (i) Let B be a hyponormal bilateral operator weighted shift, and let $r_- = \min\{r \in \sigma(A_-)\}$ and $r_+ = \max\{r \in \sigma(A_+)\}$; then

$$\sigma(U \otimes (A_- \oplus A_+)) \subset \sigma(B) \subset \{\lambda \in \mathbb{C} : r_- \leq |\lambda| \leq r_+\}$$

and $B - \lambda$ is a semi-Fredholm operator with trivial kernel and negative index for all $\lambda \in \sigma(B) \cap \rho_{\text{s-F}}(B)$. Furthermore, if this set is not empty, and

$$\alpha = \sup\{|\text{ind}(\lambda - B)| : \lambda \in \sigma(B) \cap \rho_{\text{s-F}}(B)\}$$

($1 \leq \alpha \leq \infty$), then the multiplicity of B cannot be smaller than $\alpha + 1$; that is, if $\ell^2(\mathcal{H}) = \vee\{B^k x : k \geq 0, x \in \Gamma\}$ for some subset Γ of $\ell^2(\mathcal{H})$, then $\text{card}(\Gamma) \geq \alpha + 1$.

(ii) If B is a hyponormal operator of finite multiplicity, then the self-commutator, $[B^*, B] := B^*B - BB^*$, of B is a trace class operator and $\sigma_e(B) = \sigma_{\ell \text{ re}}(B) = \sigma_{\text{red}}(B) = \sigma(U \otimes (A_- \oplus A_+))$.

Corollary 3. If B is hyponormal and $1 \leq \dim \mathcal{H} = m < \infty$, then $\sigma(B)$ is the union of m annuli of the form

$$\Delta_j = \{\lambda \in \mathbb{C} : r_j^- \leq |\lambda| \leq r_j^+\}$$

($0 \leq r_1^- \leq r_2^- \leq \dots \leq r_m^-$, $0 \leq r_1^+ \leq r_2^+ \leq \dots \leq r_m^+$, and $r_j^- \leq r_j^+$ for all $j = 1, 2, \dots, m$).

Either $B \simeq \sum_{j=1}^m r_j^+ U$ is normal (all the annuli degenerate to circles, or to a point, and B is cyclic if and only if all the radii r_j^+ are positive and distinct), or at least one of the annuli has nonempty interior and B is not cyclic. In the second case, $\sigma_e(B) = \sigma_{\ell \text{ re}}(B) = \sigma_{\text{red}}(B) = \bigcup_{j=1}^m \partial \Delta_j$; moreover, for $\lambda \in \sigma(B) \setminus \sigma_e(B)$,

$$-m \leq \text{ind}(\lambda - B) = -\text{card}\{j : \lambda \in \Delta_j\} \leq -1.$$

If $\ker B \neq \{0\}$, then $B \simeq B \oplus 0$, where 0 acts on an infinite dimensional space, and therefore B and B^* have infinite multiplicity. If B has trivial kernel, then multiplicity $(B) \leq m + 1$.

J. B. Conway has raised the following question [8, p. 166]:

Open Problem 8.24. If B is a hyponormal bilateral operator weighted shift, can B have a cyclic vector?

(In the context of Conway's book, it is tacitly assumed that B is hyponormal, but not normal.)

Theorem 2 implies that if all the weights are hermitian, then Conway's problem has a negative answer in a large number of cases. Indeed, if there is cyclic hyponormal (but not normal) bilateral operator weighted shift with positive weights, then $A_+^2 - A_-^2$ must be a trace class operator, $\sigma(A_+) = \sigma(A_-)$, and B is a compact perturbation of $U \otimes A_+$ (and several other restrictions; see Corollary 5 below).

However, if we allow nonhermitian weights, Conway's problem has an affirmative answer: *if $\mathcal{H}$ is infinite dimensional, then there exists a cyclic, nonnormal, hyponormal bilateral operator weighted shift B on $\ell^2(\mathcal{H})$.*

The example will be given in §3, which deals with the case when some of the weights do not have a polar decomposition with a unitary factor.

2. PROOF OF THE MAIN RESULTS

Suppose $r \in \sigma(A_+)$, $\varepsilon > 0$, and $x \in \mathcal{H}$ is a unit vector such that $\|(A_+ - r)x\| < \varepsilon$; then $\|(A_n - r)x\| < 2\varepsilon$ for all $n \geq n_0(\varepsilon)$. Therefore, if $\hat{x}(n) = \frac{1}{n}(\dots, 0, [0], 0, \dots, 0, x, x, \dots, x, 0, \dots)$ (the x 's are located in coordinates $n^2 + 1$ through $2n^2$), then $\hat{x}(n) \in \ell^2(\mathcal{H})$ is a unit vector, and a simple calculation shows that

$$\begin{aligned} & \| (B - r)\hat{x}(n) \| + \| (B - r)^* \hat{x}(n) \| \\ & \leq \frac{2}{n} (\|B\| + r) + \frac{1}{n} \left(\sum_{k=n^2}^{2n^2} \|(A_n - r)x\|^2 \right)^{1/2} < 3\varepsilon, \end{aligned}$$

provided $n \geq n_1 \geq n_0$.

Since $\varepsilon > 0$ can be chosen arbitrarily small, we infer that r is a reducing essential eigenvalue of B , in the sense of Salinas, and therefore $B \simeq_a B \oplus rl$.

Obviously, the same argument shows that $B \simeq_a B \oplus sl$ for each $s \in \sigma(A_-)$. Since $B \simeq e^{i\theta} B$ ($0 \leq \theta \leq 2\pi$), it follows that $B \simeq_a B \oplus N$ for some normal operator N with $\sigma(N) = \sigma_e(N) = \{\lambda \in \mathbb{C} : |\lambda| \in \sigma(A_-) \cup \sigma(A_+)\}$. Since this last set coincides with $\sigma(U \otimes [A_- \oplus A_+])$, we deduce that $N \simeq_a U \otimes (A_- \oplus A_+)$, and therefore

$$B \simeq_a B \oplus [U \otimes (A_- \oplus A_+)].$$

It readily follows that $\sigma_{\text{red}}(B)$ includes the spectrum of $U \otimes (A_- \oplus A_+)$. This proves the first statement.

Now assume that $\mathcal{H}$ is finite dimensional. Observe that

$$B = \begin{pmatrix} B_+ & F \\ 0 & B_- \end{pmatrix}_{\ell^2_+(\mathcal{H}) \oplus \ell^2_-(\mathcal{H})},$$

where

$$F = \begin{pmatrix} 0 & 0 \\ A_1 & 0 \end{pmatrix}$$

(with respect to suitable decompositions of $\ell_+^2(\mathcal{H})$ and $\ell_-^2(\mathcal{H})$ as direct sums of one copy of $\mathcal{H}$ and its orthogonal complement), so that $\text{rank } F \leq m$, and B_+ and B_- are unitarily equivalent to compact perturbations of $S \otimes A_+$, and, respectively, $S^* \otimes A_-$ (because, in this case, $\|A_{-n} - A_-\| \rightarrow 0$ and $\|A_n - A_+\| \rightarrow 0$ as $n \rightarrow \infty$!).

Hence,

$$B \simeq (S \otimes A_+) \oplus (S^* \otimes A_-) + K,$$

where $K \in \mathcal{K}$ (= the ideal of all compact operators).

Since $\pi[(S \otimes A_+) \oplus (S^* \otimes A_-)]$ is a normal element of the quotient Calkin algebra $\mathcal{L}(\ell^2(\mathcal{H}))/\mathcal{K}$ (see [7]), we conclude that

$$\begin{aligned} \sigma_e(B) &= \sigma_{\ell \text{ re}}(B) = \sigma_e(S \otimes A_+) \cup \sigma_e(S^* \otimes A_-) \\ &= \{\lambda \in \mathbb{C} : |\lambda| \in \sigma(A_-) \cup \sigma(A_+)\}. \end{aligned}$$

The proof of Theorem 1 is now complete. $\square$

The author conjectures that the conditions “ $A_{-n} \rightarrow A_-$ and $A_n \rightarrow A_+$ strongly” can be relaxed to “ $A_{-n} \rightarrow A_-$ and $A_n \rightarrow A_+$ weakly”.

In general, the inclusion $\sigma_{\ell \text{ re}}(B) \supset \{\lambda \in \mathbb{C} : |\lambda| \in \sigma(A_-) \cup \sigma(A_+)\}$ is proper, even if B is hyponormal. For instance, if $\mathcal{H} = L^2([0, \infty))$, $A_n = 1$ for $-\infty < n < 3$, and $A_n = 1 + 2P_{L^2([0, n])} + P_{L^2([n, 3n])}$ ($n \geq 3$), and $\hat{x}(n) = \frac{1}{n}(\dots, 0, [0], 0, \dots, 0, e_n, e_n, \dots, e_n, 0, \dots)$ (e_n is the characteristic function of the interval $[2n^3, 2n^3 + 1]$ and the k th coordinate is equal to e_n for $n^3 + 1 \leq k \leq n^3 + n^2$, and zero otherwise), then B is hyponormal and invertible,

$$A_{-n} \rightarrow 1 \quad \text{and} \quad A_n \rightarrow 3 \quad (\text{strongly, as } n \rightarrow \infty),$$

but

$$\|(B - 2)^* \hat{x}(n)\| \leq \|(B - 2)\hat{x}(n)\| < \frac{2}{n} \rightarrow 0 \quad (n \rightarrow \infty),$$

whence we deduce that $B \simeq_a B \oplus 2U$, although $2 \notin \sigma(A_-) \cup \sigma(A_+) = \{1, 3\}$.

If $\|A_{-n} - A_-\| + \|A_n - A_+\| \rightarrow 0$ ($n \rightarrow \infty$), then we can prove that $\sigma_{\text{red}}(B) \supset \sigma(U \otimes (A_- \oplus A_+))$ by a more direct argument, by using Berg's technique [3], as described in [14, Chapter 2].

Conjecture 4. If $\|A_{-n} - A_-\| + \|A_n - A_+\| \rightarrow 0$ ($n \rightarrow \infty$), then $\sigma_{\ell \text{ re}}(B) = \sigma_{\text{red}}(B) = \sigma(U \otimes (A_- \oplus A_+))$.

Proof of Theorem 2. (i) The first inclusion follows from Theorem 1, and the second one follows from the fact that $\|B\| = \text{sp}(B)$ for all hyponormal operators (see Introduction; [16]).

If B is hyponormal and μ is an eigenvalue of B , then $B \simeq B_\mu \oplus \mu I_{\mathbb{C}}$ [16]. Since $B \simeq e^{i\theta} B$, the same argument applies to every point of the circle of radius

$|\mu|$. The separability of $\mathcal{H}$ makes this impossible, unless $\mu = 0$. In this case, $\ker A_p \neq \{0\}$ for some p . Whence it readily follows that $\ker A_p \subset \ker A_n \neq \{0\}$ for all $n \geq p$; and therefore $B \simeq B \oplus 0$, where 0 acts on an infinite dimensional space (and therefore B and B^* have infinite multiplicity; clearly, neither B nor B^* can be a semi-Fredholm operator).

Thus, if $\lambda \in \sigma(B) \cap \rho_{\text{s-F}}(B)$, then $\ker(\lambda - B) = \{0\}$ and $-\infty \leq -h = \text{ind}(\lambda - B) \leq -1$. But B cannot be a semi-Fredholm operator of negative index. Thus, if Ω is a component of

$$\{\lambda \in \rho_{\text{s-F}}(B) : \text{ind}(\lambda - B) = -h\},$$

then Ω is either an annulus or a punctured disk. In either case, Ω is not simply connected, and therefore (using [13, Theorem 1] or [15])

$$\text{multiplicity}(B) \geq h + 1,$$

whence we obtain the last statement, by taking supremum over all possible λ 's.

(ii) If B is hyponormal and has finite multiplicity, the Berger-Shaw trace inequality implies that $[B^*, B]$ is a trace class operator; this is equivalent to saying that

$$A_+^2 - A_-^2 \text{ is trace class}$$

(trace norm $(A_+^2 - A_-^2) = \text{trace}[B^*, B] \leq \frac{1}{\pi} \text{area } \sigma(B) \times \text{multiplicity}(B)$; see [4, 5]), and therefore compact. In particular, $\sigma_e(A_+^2) = \sigma_e(A_-^2)$, whence we obtain $\sigma_e(A_+) = \sigma_e(A_-)$. (The Berger-Shaw trace inequality applies to hyponormal operators of finite *rational* multiplicity. But the difference is irrelevant here, because $\text{rat.multiplicity}(T) \leq \text{multiplicity}(T) \leq \text{rat.multiplicity}(T) + 1$ for every operator T [13, Proposition 2].)

If A_- is bounded below by $\delta > 0$ and γ is the (positively oriented) circle of radius $\|B\|^2 - (\delta/2)^2$ centered at $\|B\|^2$, then by using the second resolvent equation [11], [17], we see that

$$\begin{aligned} A_+ - A_- &= \frac{1}{2\pi i} \int_{\gamma} \sqrt{\lambda} \{(\lambda - A_+^2)^{-1} - (\lambda - A_-^2)^{-1}\} d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma} \sqrt{\lambda} (\lambda - A_+^2)^{-1} (A_+^2 - A_-^2) (\lambda - A_-^2)^{-1} d\lambda \end{aligned}$$

is also a trace class operator.

If A_- is not bounded below, then

$$\|A_- - (A_-^2 + \varepsilon l)^{1/2}\| \rightarrow 0, \quad \|A_+ - (A_+^2 + \varepsilon l)^{1/2}\| \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

and

$$\|A_+ - A_- - [(A_+^2 + \varepsilon l)^{1/2} - (A_-^2 + \varepsilon l)^{1/2}]\| \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

whence we infer that $A_+ - A_-$ is a *positive hermitian compact operator*. (It is not difficult to check that we also have $\|A_{-n} - A_-\| + \|A_n - A_+\| \rightarrow 0$, as $n \rightarrow \infty$.)

Since $[(B \oplus B^*)^*, B \oplus B^*] = [B^*, B] \oplus [B, B^*]$ is compact and (by using the invariance under rotation of every part of the spectrum of $\sigma(B)$) $\text{ind}(\lambda - B \oplus B^*) = 0$ for all $\lambda \in \rho_{\text{s-F}}(B) = \rho_{\text{s-F}}(B^*)$, the Brown–Douglas–Fillmore theorem implies that $B \oplus B^*$ is a compact perturbation of a normal operator M , and $\sigma_e(B) = \sigma_{\text{red}}(B) = \sigma_e(B^*) = \sigma_{\text{red}}(B^*) = \sigma(M) = \sigma_e(M)$ [6].

Observe that a suitable compact perturbation of B can be written as an operator matrix (C_{ij}) ($-\infty < i, j < \infty$) with respect to a decomposition $\ell^2(\mathcal{H}) = \sum \oplus \{\mathcal{H}_j: -\infty < j < \infty; \mathcal{H}_j \simeq \mathcal{H} \text{ for all } j\}$, where $C_{i,i+1} = A_+$, or A_- , according to whether $i \geq 0$ or $i < 0$, and $C_{ij} = 0$ if $j \neq i+1$. Similarly, B^* can be written as an operator matrix of exactly the same form, with $C_{i,i+1} = A_-$, or A_+ , according to whether $i \geq 0$, or $i < 0$. Given $\varepsilon > 0$, we can use Berg’s technique [3; 14, Chapter 2] in order to approximate $B \oplus B^*$ in less than ε by an operator $T \simeq R \oplus V$, where $R \simeq U \otimes A_-$ and V is a bilateral operator weighted shift such that all but finitely many weights are equal to A_+ and all the weights satisfy $A_- \leq V_n \leq A_+$. Clearly, T is unitarily equivalent to a compact perturbation of $U \otimes (A_- \oplus A_+)$, and therefore $\sigma_e(T) = \sigma_{\text{red}}(T) = \sigma_e(U \otimes (A_- \oplus A_+))$.

Since $B \oplus B^*$ and its approximation T project into normal elements of the Calkin algebra, and since $a \rightarrow \sigma(a)$ is always a continuous mapping, when restricted to normal elements of any C^* -algebra, we conclude that $\sigma_e(B) = \sigma_e(B \oplus B^*) = \sigma_e(U \otimes (A_- \oplus A_+)) = \sigma(U \otimes (A_- \oplus A_+))$. $\square$

Let B be as in Theorem 2 (ii), so that $\sigma_e(B)$ is the union of $\sigma(U \otimes A_-)$ and $\sigma(U \otimes A_+)$. Recall that $A_+ - A_-$ is compact, so that $\sigma(A_+)$ and $\sigma(A_-)$ have exactly the same nonisolated points. Assume that $r \in \sigma(A_-) \setminus \sigma(A_+)$; then r is an isolated point of $\sigma(A_-)$, and therefore the circle of radius r is a clopen subset of $\sigma_e(B)$. If this circle is also a clopen subset of $\sigma(B)$, then the hyponormal operator B is unitarily equivalent to $rU \oplus B'$, for some hyponormal operator B' with $\sigma(B') = \sigma(B) \setminus \sigma(rU)$ [16]. But this would imply that $r \in \sigma(A_+)$, a contradiction.

Hence, the circle of radius r cannot be a clopen subset of $\sigma(B)$. It readily follows that if $\varepsilon > 0$ is small enough, then either $(r - \varepsilon) - B$, or $(r + \varepsilon) - B$ is a semi-Fredholm operator of negative index, whence we infer that B cannot be cyclic.

The same argument shows that B cannot be cyclic if $\sigma(A_+)$ is not included in $\sigma(A_-)$.

Furthermore, (even if $\sigma(B) = \sigma_e(B)$) if B^* has a nonzero eigenvalue, by proceeding along the lines of [12], we can still conclude that B cannot be a cyclic operator.

Hence, we have obtained the following.

Corollary 5. *If B is a cyclic, hyponormal, bilateral operator weighted shift with positive weights $\{A_n\}$, then*

- (i) B^* has no eigenvalues,

- (ii) $A_+^2 - A_-^2$ is a positive hermitian operator whose trace norm does not exceed $\frac{1}{\pi} \text{area } \sigma(B)$,
- (iii) $\sigma(A_+) = \sigma(A_-)$ and $\sigma(B) = \sigma_e(B) = \sigma_{\ell \text{ re}}(B) = \sigma_{\text{red}}(B) = \sigma(U \otimes A_+) = \sigma(U \otimes A_-)$, and
- (iv) $\ker(r - A_-) = \ker(r - A_+)$ and this subspace has dimension one, for each isolated point r of $\sigma(A_+)$.

The last statement follows from the fact that $U \oplus U$ is not cyclic. The author strongly suspects that if B is cyclic, then it is necessarily a normal operator. (In this case, B has the form $B = U \otimes A$ for some cyclic positive hermitian operator A with trivial kernel.)

Proof of Corollary 3. The spectral structure, and most of the other statements, follow immediately from Theorems 1 and 2 (ii).

The only point not contained in the theorems is multiplicity. Assume that $\ker B$ is trivial; then all the weights are invertible and B_- has a cyclic vector $x_- \in \ell^2(\mathcal{H})$. (This follows from the same argument that shows that $(S^* \otimes 1)$ has a cyclic vector; see, e.g., [11].) Now it is straightforward to check that if $\{e_1, e_2, \dots, e_m\}$ is a basis of $\mathcal{H}$ and $\hat{y}_j = (\dots, 0, 0, [e_j], 0, 0, \dots)$ ($j = 1, 2, \dots, m$), then

$$\ell^2(\mathcal{H}) = \vee \{B^k(x_- \oplus 0), B^k y_j : k \geq 0, 1 \leq j \leq m\}.$$

Therefore, multiplicity $(B) \leq m + 1$. $\square$

3. THE CASE WHEN THE WEIGHTS DO NOT HAVE A GOOD FACTORIZATION

Throughout this section we shall assume that $\mathcal{H}$ is infinite dimensional, B is hyponormal, and some of the weights (say, A_0) cannot be factored as Unitary $\times$ Positive; then either $\ker A_0$ is nontrivial, or $\ker A_0^*$ is nontrivial, $A_{-1}A_{-1}^* \leq A_0^*A_0$ and $A_0A_0^* \leq A_{-1}^*A_{-1}$.

If $\ker A_0 \neq \{0\}$, then $\ker B^* \supset \ker B \neq \{0\}$ and $B \simeq B' \oplus 0_{\ker B}$. If $\ker B$ is infinite dimensional, then B and B^* have infinite multiplicity; if B is cyclic, then $\ker B$ must be, at most, one dimensional, and this implies that $\ker A_n = \{0\}$ for all $n \neq 0$. For example, if $A_0 = S^*$, $A_{-1} = S$, and $A_n = 1$ for all $n \neq 0, -1$, then $B \simeq (U \otimes 1) \oplus 0_{\mathbb{C}}$ (normal, not cyclic). If H is positive hermitian, cyclic, and $\ker H$ is trivial, and we choose $A_0 = HS^*$, $A_{-1} = SH$, and $A_n = H$ for all $n \neq 0, -1$ then

$$B \simeq (U \otimes H) \oplus 0_{\mathbb{C}} \text{ (normal and cyclic).}$$

Example 6. Define B as usual, with $A_{-n} = J$ ($n > 0$), $A_0 = SJ$ and

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}_{(\ker S^*)^\perp}^{(\ker S^*)} \quad (n > 0),$$

where J is the diagonal hermitian operator with (simple) eigenvalues $2, 3/2, 4/3, 5/4, \dots$. Then B is a cyclic, nonnormal, hyponormal bilateral operator weighted shift; moreover, $B \simeq \{\sum \bigoplus_{n=1}^{\infty} (1 + \frac{1}{n})U\} \oplus S$.

Proof. Write B as an upper triangular matrix with respect to the decomposition $\ell^2(\mathcal{H}) \simeq \sum \oplus \{\mathcal{H}_k : -\infty < k < \infty\}$. A straightforward analysis shows that

$$B \simeq \left\{ \sum \bigoplus_{n=1}^{\infty} \left(1 + \frac{1}{n} \right) U \right\} \oplus S,$$

which is obviously hyponormal, but not normal.

It only remains to show that B is cyclic. Think of U as the “multiplication by $e^{i\theta}$ ” on $L^2(\partial\mathbb{D}, dm)$ (dm = normalized linear Lebesgue measure on the unit circle), and S is the unilateral shift with respect to the orthonormal basis $\{e_j = \lambda^j\}_{j=0}^{\infty}$ of $H^2(\partial\mathbb{D})$; then B is the operator “multiplication by λ ” on

$$\left\{ \sum \bigoplus_{n=1}^{\infty} L^2(\Delta_n, dm_n) \right\} \oplus H^2(\partial\mathbb{D}),$$

where Δ_n is the circle of radius $1 + \frac{1}{n}$ and dm_n is the normalized linear Lebesgue measure on this circle ($n = 1, 2, \dots$). Let Δ_n^+ , Δ_n^- denote the intersection of Δ_n with the upper half-plane and, respectively, with the lower half-plane. By Runge’s theorem [10], there exist sequences of polynomials $\{p_{nk}\}_{k=1}^{\infty}$ ($n = 1, 2, \dots$) such that

$$|p_{nk}(\lambda) - 1| < 1/k \text{ for all } \lambda \in \Delta_n^+$$

and

$$|p_{nk}(\lambda)| < 1/k \text{ for } \lambda \in \Delta_{n+1} \cup \left\{ \zeta \in \Delta_n^- : -\pi + \frac{1}{k} < \arg \zeta < -\frac{1}{k} \right\}$$

(so that $p_{nk}(\lambda) \rightarrow 0$, as $k \rightarrow \infty$, uniformly on the closed disk of radius $1 + 1/(n+1)$, $n = 1, 2, \dots$).

Now we use an argument very close to [13, Example 5, pp. 75–76]. Let $f_n \in L^2(\Delta_n, dm_n)$ be a function such that $f_n \equiv 1$ on Δ_n^+ , $0 < f(\lambda) \leq 1$, f is continuous on Δ_n^- and approaches to 0 when λ approaches the real axis from below, and

$$\int_{\Delta_n} |g_n(\lambda) - p_{nk}(\lambda)f_n(\lambda)|^2 dm_n < 2/k^2 \quad (k = 1, 2, \dots),$$

where g_n denotes the characteristic function of Δ_n^+ ($n = 1, 2, \dots$).

Finally, define

$$y = \left\{ \sum \bigoplus_{n=1}^{\infty} 2^{-n} f_n \right\} \oplus e_0.$$

Let $\mathcal{M}$ be the closed linear span of $\{B^k y\}_{k=0}^{\infty}$. Clearly,

$$g_1 = (\text{norm}) \lim_{k \rightarrow \infty} 2p_{1k}(B)y \in \mathcal{M},$$

whence we deduce that $L^2(\Delta_1^+, dm_1) \oplus \{0\} = \bigvee \{B^k g_1\}_{k=0}^{\infty} \subset \mathcal{M}$; moreover, since $f_1 - g_1$ is a continuous function, strictly positive on Δ_n^- , a new application

of Runge's theorem shows that $f_1 - g_1 \in \mathcal{M}$, and therefore

$$L^2(\Delta_1, dm_1) \oplus \{0\} = L^2(\Delta_1^+, dm_1) \oplus L^2(\Delta_1^-, dm_1) \oplus \{0\} \subset \mathcal{M}.$$

A formal repetition of the same argument (with y replaced by $y_1 = y - 2^{-1}f_1 \oplus 0$) indicates that $L^2(\Delta_2, dm_2) \oplus \{0\}$ is also included in $\mathcal{M}$, and therefore

$$L^2(\Delta_1, dm_1) \oplus L^2(\Delta_2, dm_2) \oplus \{0\} \subset \mathcal{M}.$$

By induction, we deduce that $\{\sum \bigoplus_{n=1}^{\infty} L^2(\Delta_n, dm_n)\} \oplus \{0\} \subset \mathcal{M}$. Therefore,

$$0 \oplus e_0 = y - \left\{ \sum \bigoplus_{n=1}^{\infty} 2^{-n} f_n \right\} \oplus 0 \in \mathcal{M},$$

whence we conclude that $\{0\} \oplus H^2(\partial \mathbb{D}) \subset \mathcal{M}$.

Hence, y is a cyclic vector for B . $\square$

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DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287