THE HAAGERUP TYPE CROSS NORM ON C^* -ALGEBRAS

TAKASHI ITOH

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Several cross norms between the projective C^* -norm and the projective cross norm are introduced. Let A and B be C^* -algebras. It is shown that $\| \cdot \|_h$ is equivalent to $\| \cdot \|_\gamma$ on $A \otimes B$ if and only if A or B is subhomogeneous.

INTRODUCTION

In [4] Effros and Kishimoto introduced a norm on the tensor product $A \otimes B$ of C^* -algebras as follows:

$$\|x\|_h = \inf \left\{ \left\| \sum_{i=1}^n a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n b_i^* b_i \right\|^{\frac{1}{2}} \mid x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B \right\}$$

for any $x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B$, which was called the Haagerup norm.

Let $\| \cdot \|_\gamma$ be the projective cross norm on $A \otimes B$, namely

$$\|x\|_\gamma = \inf \left\{ \sum_{i=1}^n \|a_i\| \|b_i\| \mid x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B \right\}$$

for any $x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B$.

On the tensor product of commutative C^* -algebras, $\| \cdot \|_h$ and $\| \cdot \|_\gamma$ are equivalent [2], indeed

$$\| \cdot \|_h \leq \| \cdot \|_\gamma \leq K_G^C \| \cdot \|_h,$$

where K_G^C is the complex Grothendieck constant.

In this paper, we introduce several Haagerup type cross norms which are not equivalent with each other, however, if the two C^* -algebras are commutative, they coincide. One of the aims of this paper is to recognize noncommutativity from the view point of the noncommutative Grothendieck inequality [5]. To see the difference of these norms, we determine conditions which are equivalent.

Received by the editors February 23, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46L05.

Key words and phrases. Haagerup norm.

As a consequence, we get that the following two conditions are equivalent in C^* -algebras A and B :

- (1) $\|\cdot\|_h$ is equivalent to $\|\cdot\|_\gamma$ on $A \otimes B$,
- (2) A or B is subhomogeneous.

By this and (cf. [1, Theorem 6.1.5; 8]), we conclude that $\|\cdot\|_h$ is strictly in between $\|\cdot\|_{\max}$ and $\|\cdot\|_\gamma$. These strengthen the result of Huruya's [6].

RESULTS

Let A and B be C^* -algebras. We denote the set of all $n \times n$ matrices on the complex number field $\mathbb{C}$ by M_n and let $B(H)$ be all bounded operators on a Hilbert space H . It has been shown in [8] that $\|\cdot\|_h$ has another representation, which will be written in the table. We introduce several cross norms as follows

Let $\alpha \in \{\max, \max', h, h', 'h', 'h', J\}$ and define

$$\|x\|_\alpha = \sup \left\{ \left\| \sum_{i=1}^n \pi_1(a_i) \pi_2(b_i) \right\| \mid \begin{array}{l} \pi_1 \text{ and } \pi_2 \text{ run through} \\ \text{all representations of } A \\ \text{and } B \text{ which have the} \\ \text{conditions in the table,} \\ \text{respectively} \end{array} \right\}$$

for any $x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B$. Since the map $\pi_1 \times \pi_2(a, b) = \pi_1(a) \pi_2(b)$ is bilinear, $\|x\|_\alpha$ is independent of the decomposition $x = \sum_{i=1}^n a_i \otimes b_i$.

	max	max'	h	h'	'h	'h'	J
π_1	(i) (iv)	(i) (iv)	(i)	(i)	(ii)	(ii)	(iii)
π_2	(i) (iv)	(ii) (iv)	(i)	(ii)	(i)	(ii)	(iii)
and	(v)	(v)					

where

- (i) is a $*$ -representation,
- (ii) is an anti $*$ -representation,
- (iii) is a Jordan $*$ -representation, which is the direct sum of a $*$ -representation and an anti $*$ -representation,
- (iv) is non-degenerate,
- (v) is: $\pi_1(a) \pi_2(b) = \pi_2(b) \pi_1(a)$ for all $a \in A$ and all $b \in B$.

It is easy to see that $\|\cdot\|_\alpha$ satisfies the cross norm conditions except that it has no null space and has the condition $\|a \otimes b\|_\alpha = \|a\| \|b\|$ for any $a \in A$, $b \in B$. However it is well known that $\|\cdot\|_\lambda \leq \|\cdot\|_{\max}$ [cf. 13], where $\|\cdot\|_\lambda$ is the injective cross norm. We can also prove that $\|\cdot\|_\lambda \leq \|\cdot\|_{\max'}$ by the similar way. Hence the following asserts that $\|\cdot\|_\alpha$ defines a cross norm.

Proposition 1. *Let*

$$(\alpha, \beta) \in \{(\max, h), (\max, 'h'), (\max', h'), (\max', 'h')\}.$$

Then

$$\|x\|_\alpha \leq \|x\|_\beta \leq \|x\|_J$$

for any $x \in A \otimes B$.

Proof. Suppose that $(\alpha, \beta) = (\max, 'h')$. Let θ be the transpose map on $B(H)$ with respect to some orthonormal basis and π_1 (resp. π_2) be an anti $*$ -representation of A (resp. B) on $B(H)$. We have only to observe that $\theta(\pi_i(\cdot))$ is a $*$ -representation and

$$\left\| \sum_{i=1}^n \theta(\pi_2(b_i)) \theta(\pi_1(a_i)) \right\| = \left\| \sum_{i=1}^n \pi_1(a_i) \pi_2(b_i) \right\|$$

for any $x = \sum_{i=1}^n a_i \otimes b_i$. The rest is trivial.

Remark 2. Let $\alpha \in \{h, h', 'h, 'h', J\}$.

(a) We can choose suitable π_1 and π_2 which attain the norm: namely, for any $x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B$, there exist π_1 and π_2 such that

$$\|x\|_\alpha = \left\| \sum_{i=1}^n \pi_1(a_i) \pi_2(b_i) \right\|,$$

where π_1 and π_2 satisfy the conditions in the definition of $\|\cdot\|_\alpha$ [8].

(b) $\|\cdot\|_\alpha$ has another representation. For example,

$$\|x\|_{h'} = \inf \left\{ \left\| \sum_{i=1}^n a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n b_i b_i^* \right\|^{\frac{1}{2}} \mid x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B \right\}.$$

The following lemma of R. R. Smith in [12] (the non unital version is due to T. Huruya and J. Tomiyama [7]) is useful in determining noncommutativity.

Lemma 3 [2.7]. *Suppose that A is not a subhomogeneous C^* -algebra. Then, given n and $\epsilon > 0$, there exist completely positive contractions $\rho_A: M_n \rightarrow A$ and $\sigma_A: A \rightarrow M_n$, such that $\|\sigma_A \rho_A - id\| < \epsilon$ on M_n .*

We determine the pairwise difference of the norms $\|\cdot\|_h$, $\|\cdot\|_{h'}$, $\|\cdot\|_{\cdot h}$ and $\|\cdot\|_{h'}$. Two ways of representing each norm will be needed.

Theorem 4. *Let A and B be C^* -algebras and let*

$$(\alpha, \beta) \in \{(h, h'), (h, 'h'), (h, 'h'), (h', 'h'), (h', 'h'), ('h, 'h')\}.$$

Then the following two conditions are equivalent:

- (1) $\|\cdot\|_\alpha$ is equivalent to $\|\cdot\|_\beta$ on $A \otimes B$,
- (2) A or B is subhomogeneous.

Proof. There are six cases, but it suffices to show only one case; namely $(\alpha, \beta) = (h, h')$ since the other cases can be shown in the same way.

(1) $\Rightarrow$ (2) Suppose that neither A nor B is subhomogeneous. Given n and $\epsilon > 0$, there exist four completely positive contractions ρ_A, σ_A and ρ_B, σ_B which satisfy the conditions in Lemma 3. Let $\{e_{ij}\}_{i,j=1}^n$ be the matrix units in M_n and let

$$\rho_A(e_{1i}) = a_i, \quad \rho_B(e_{1i}) = b_i \quad \text{for } i = 1, \dots, n.$$

It is easy to see that $\theta_n(\sigma_B(\cdot))$ is completely copositive [14], where θ_n is the transpose map on M_n . Then there are minimal Stinespring representations $\{\pi_1, H_1, V_1\}$ of σ_A and $\{\pi_2, H_2, V_2\}$ of σ_B , where π_1 is a $*$ -representation on H_1 of A and π_2 is an anti $*$ -representation on H_2 of B . Put $x_n = \sum_{i=1}^n a_i \otimes b_i \in A \otimes B$. By using a unitary dilation

$$U = \begin{bmatrix} \sqrt{1 - V_1 V_2^* V_2 V_1^*} & V_1 V_2^* \\ -V_2 V_1^* & \sqrt{1 - V_2 V_1^* V_1 V_2^*} \end{bmatrix} \quad \text{of } V_1 V_2^*,$$

it follows that

$$\begin{aligned} \|x_n\|_h &\geq \left\| \sum_{i=1}^n U^* \begin{bmatrix} \pi_1(a_i) & 0 \\ 0 & 0 \end{bmatrix} U \begin{bmatrix} 0 & 0 \\ 0 & \pi_2(b_i) \end{bmatrix} \right\| \\ &\geq \|V_1\| \left\| \sum_{i=1}^n \pi_1(a_i) V_1 V_2^* \pi_2(b_i) \right\| \|V_2\| \\ &\geq \left\| \sum_{i=1}^n \sigma_A(a_i) \theta_n(\sigma_B(b_i)) \right\|. \end{aligned}$$

Since

$$\|\sigma_A(a_i) - e_{1i}\| < \epsilon \quad \text{and} \quad \|\theta_n(\sigma_B(b_i)) - e_{i1}\| < \epsilon,$$

we have

$$\begin{aligned} &\left\| \sum_{i=1}^n \sigma_A(a_i) \theta_n(\sigma_B(b_i)) \right\| - \left\| \sum_{i=1}^n e_{1i} e_{i1} \right\| \\ &\leq \sum_{i=1}^n \|\sigma_A(a_i)\| \|\theta_n(\sigma_B(b_i)) - e_{i1}\| + \sum_{i=1}^n \|\sigma_A(a_i) - e_{1i}\| \|e_{i1}\| \\ &< 2n\epsilon. \end{aligned}$$

Hence

$$\begin{aligned} \|x_n\|_h &\geq \left\| \sum_{i=1}^n e_{1i} e_{i1} \right\| - 2n\epsilon \\ &= n(1 - 2\epsilon). \end{aligned}$$

Since we can take n and ϵ independently, we have $\|x_n\|_h \geq n$.

On the other hand, since ρ_A and ρ_B satisfy the Schwarz inequality, we have

$$\begin{aligned} \|x_n\|_h &\leq \left\| \sum_{i=1}^n a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n b_i^* b_i \right\|^{\frac{1}{2}} \\ &\leq \left\| \sum_{i=1}^n \rho_A(e_{1i} e_{1i}^*) \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n \rho_B(e_{1i}^* e_{1i}) \right\|^{\frac{1}{2}} \\ &\leq \left\| \sum_{i=1}^n e_{1i} e_{i1} \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n e_{i1} e_{1i} \right\|^{\frac{1}{2}} \\ &= \sqrt{n}. \end{aligned}$$

This is a contradiction.

(2) $\Rightarrow$ (1) Suppose that A is n -subhomogeneous for some n . Since the enveloping von Neumann algebra of A is of type I with bounded degree less than n [10] and the part of the direct sum is negligible in this proof, we may assume that A is contained in $Z \otimes M_n$ where Z is a commutative von Neumann algebra. Let $x = \sum_{k=1}^m a_k \otimes b_k \in A \otimes B$. The a_k can be rewritten as $a_k = \sum_{i,j=1}^n a_k(i, j) \otimes e_{ij}$, where $a_k(i, j) \in Z$. As mentioned in Remark 2, there exist a $*$ -representation π_1 of A and an anti $*$ -representation π'_2 of B acting on the same Hilbert space H such that

$$\|x\|_h = \left\| \sum_{k=1}^m \pi_1(a_k) \pi'_2(b_k) \right\|.$$

We can extend π_1 as a $*$ -representation of $Z \otimes M_n$ on a larger Hilbert space K (cf. [3, Proposition 2.10.2]). Since we allow that the representation is degenerate, we consider that π'_2 is an anti $*$ -representation of B on K . Let Ω be the pure state space of the commutative C^* -algebra $\pi_1(Z \otimes 1_n)$. Then we have

$$\begin{aligned} \|x\|_h &\leq \sum_{i,j=1}^n \|\pi_1(1 \otimes e_{ij})\| \left\| \sum_{k=1}^m \pi_1(a_k(i, j) \otimes 1_n) \pi'_2(b_k) \right\| \\ &\leq \sum_{i,j=1}^n \sup_{\omega \in \Omega} \|\pi'_2\| \left\| \sum_{k=1}^m \omega(\pi_1(a_k(i, j) \otimes 1_n)) b_k \right\| \\ &\leq \sum_{i,j=1}^n \sup_{\omega \in \Omega} \sup_{\pi_2} \left\| \sum_{k=1}^m \omega(\pi_1(a_k(i, j) \otimes 1_n)) \pi_2(b_k) \right\|, \end{aligned}$$

where π_2 runs through all $*$ -representations of B .

Now we define a $*$ -representation $\bar{\pi}_1$ of A and a $*$ -representation $\bar{\pi}_2$ of B to $B(H) \otimes M_n$ as follows:

$$\begin{aligned} \bar{\pi}_1(a_k) &= \sum_{i,j=1}^n \pi_1(a_k(i, j) \otimes 1_n) \otimes e_{ij} \\ \bar{\pi}_2(b_k) &= \pi_2(b_k) \otimes 1_n. \end{aligned}$$

Then

$$\|x\|_h \leq n^2 \sup_{\pi_1, \pi_2} \left\| \sum_{k=1}^m \bar{\pi}_1(a_k) \bar{\pi}_2(b_k) \right\|.$$

Hence we have $\|x\|_h \leq n^2 \|x\|_h$. By using a similar step, we also have $\|x\|_h \leq n^2 \|x\|_h$.

Suppose B is n -subhomogeneous. This case is easier than the one above and we have the same inequalities.

Remark 5. D. P. Blecher has already obtained the following inequality:

$$\|x\|_J \leq \|x\|_\gamma \leq 2\|x\|_J$$

for any $x \in A \otimes B[1]$. He also mentions that an example of U. Haagerup leads to the best possible constant, 2, in the inequality [cf. 1, Proposition 4.2.7]. In his thesis, $\|\cdot\|_J$ is written as $\|\cdot\|_{sh}$.

Now we get the following relationship between $\|\cdot\|_h$ and $\|\cdot\|_\gamma$.

Corollary 6. *Let A and B be C^* -algebras and let $\alpha \in \{h, h', 'h, 'h\}$. Then the following two conditions are equivalent:*

- (1) $\|\cdot\|_\alpha$ is equivalent to $\|\cdot\|_\gamma$ on $A \otimes B$,
- (2) A or B is subhomogeneous.

Proof. (1) $\Rightarrow$ (2) Suppose that neither A nor B is subhomogeneous, then there exist a sequence $\{x_n\} \subset A \otimes B$ and $\beta \in \{h, h', 'h, 'h\}$, $\beta \neq \alpha$ such that

$$\begin{aligned}\|x_n\|_\alpha &\leq 1 \\ \|x_n\|_\gamma &\geq \|x_n\|_J \geq \|x_n\|_\beta \geq n\end{aligned}$$

by Theorem 4, Proposition 1 and Remark 5.

(2) $\Rightarrow$ (1) By Theorem 4, $\|\cdot\|_h$, $\|\cdot\|_{h'}$, $\|\cdot\|_{'h}$ and $\|\cdot\|_{'h'}$ are mutually equivalent. From the definition of $\|\cdot\|_J$ and Remark 5, it follows that

$$\|\cdot\|_\gamma \leq 2\|\cdot\|_J \leq 2(\|\cdot\|_h + \|\cdot\|_{h'} + \|\cdot\|_{'h} + \|\cdot\|_{'h'}).$$

Hence there exists a constant $C > 0$ such that $\|\cdot\|_\gamma \leq C\|\cdot\|_\alpha$ for any $\alpha \in \{h, h', 'h, 'h\}$.

ACKNOWLEDGMENT

The author wishes to thank U. Haagerup for many fruitful discussions and comments. This work was done while the author was staying at the Mittag-Leffler Institute, which he wants to thank for its support and kind hospitality.

REFERENCES

1. D. P. Blecher, *Geometry of the tensor product of C^* -algebras*, Ph.D. thesis, University of Edinburgh, May 1988.
2. E. Christensen and A. M. Sinclair, *Representations of completely bounded multilinear operators*, J. Funct. Anal. **72** (1987), 151–181.
3. J. Dixmier, *C^* -algebras*, North-Holland, Amsterdam, 1979.
4. E. G. Effros and A. Kishimoto, *Module maps and Hochschild–Johnson cohomology*, Indiana Univ. Math. J. **36** (1987), 257–276.
5. U. Haagerup, *The Grothendieck inequality for bilinear forms on C^* -algebras*, Adv. in Math. **56** (1985), 93–116.
6. T. Huruya, *On compact completely bounded maps of C^* -algebras*, Michigan Math. J. **30** (1983), 213–220.
7. T. Huruya and J. Tomiyama, *Completely bounded maps of C^* -algebras*, J. Operator Theory **10** (1983), 141–152.
8. T. Itoh, *The maximal C^* -norm and the Haagerup norm*, preprint.
9. S. Kaijser and A. M. Sinclair, *Projective tensor products of C^* -algebras*, Math. Scand. **55** (1984), 161–187.
10. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.

11. V. I. Paulsen and R. R. Smith, *Multilinear maps and tensor norms on operator systems*, J. Funct. Anal. **73** (1987), 258–276.
12. R. R. Smith, *Completely bounded maps between C^* -algebras*, J. London Math. Soc. **27** (1983), 157–166.
13. M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, Heidelberg, 1979.
14. S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys. **10** (1976), 165–183.

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, RYUKYU UNIVERSITY, OKINAWA, JAPAN