HALL SUBGROUP NORMALIZERS AND CHARACTER CORRESPONDENCES IN M -GROUPS

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(Communicated by Warren J. Wong)

ABSTRACT. Let H be a Hall subgroup of an M -group G , and let N be its normalizer in G . It is shown that the group N/H' is an M -group. This involves the construction of an explicit bijection between the sets of irreducible characters of G and of N with degrees coprime to the order of H .

1. INTRODUCTION

Suppose G is a finite M -group: a group in which every irreducible complex character $\chi \in \text{Irr}(G)$ can be written in the form $\chi = \lambda^G$ for some linear character λ of some subgroup of G . While it is far from true that every subgroup of G must be an M -group, there is some evidence that certain types of subgroups might necessarily be M -groups. It has been conjectured, for instance, that Hall subgroups of M -groups inherit the property. Perhaps it is even true that if H is a Hall subgroup of G , then $N = N_G(H)$ is automatically an M -group. (This would imply that H is an M -group since normal Hall subgroups of M -groups are known to be M -groups.)

In this paper, we obtain a much more modest result. If G , H , and N are as above, and H happens to be Abelian, then N is an M -group. In fact, we get the following.

Theorem A. $H \subseteq G$, where G is an M -group and H is a Hall subgroup. Then N/H' is an M -group, where $N = N_G(H)$.

In above situation, it is easy to see that the irreducible characters of N/H' correspond precisely to those irreducible characters of N with degrees coprime to $|H|$. Theorem A essentially says, therefore, that all of these characters are monomial.

In fact, we do not quite need G to be an M -group. For any set π of primes and any group G , let us write $\text{Irr}^\pi(G)$ to denote the subset of $\text{Irr}(G)$ consisting of characters having π' -degree.

Received by the editors September 20, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20C15.

This research was partially supported by a grant from the National Science Foundation.

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 0002-9939/90 \$1.00 + \$.25 per page

Theorem B. *Let G be solvable, let π be a set of primes, and write $N = N_G(H)$, where H is a Hall π -subgroup of G . If every character in $\text{Irr}^\pi(G)$ is monomial, then every character in $\text{Irr}^\pi(N)$ is also monomial.*

Since M -groups are solvable, Theorem A is an immediate consequence of Theorem B.

By a result of T. R. Wolf [5], if G , π , and N are as in Theorem B, then (without any monomiality assumption) we have $|\text{Irr}^\pi(G)| = |\text{Irr}^\pi(N)|$. (See also [2, Theorem 2.9].) Wolf's result is crucial in our proof of Theorem B and in return, our argument strengthens Wolf's theorem by constructing a canonical bijection $\text{Irr}^\pi(G) \rightarrow \text{Irr}^\pi(N)$ in the case that $\text{Irr}^\pi(G)$ consists entirely of monomial characters.

Theorem C. *Let (G) , π , and N be as in Theorem B and suppose every $\chi \in \text{Irr}^\pi(G)$ is monomial. Then there exists a bijection $\beta: \text{Irr}^\pi(G) \rightarrow \text{Irr}^\pi(N)$ such that if $\chi \in \text{Irr}^\pi(G)$, then $\beta(\chi) = (\lambda_{N \cap K})^N$ for every choice of a pair (K, λ) with linear $\lambda \in \text{Irr}(K)$, such that $\lambda^G = \chi$ and $H \subseteq K \subseteq G$. Furthermore, for every $\chi \in \text{Irr}^\pi(G)$, there is such a pair (K, λ) .*

Of course, from the statement of Theorem C, it is immediate that the bijection β is unique. It follows, for instance, that if σ is any automorphism of the complex numbers, then $\beta(\chi^\sigma) = \beta(\chi)^\sigma$, and thus the fields $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\beta(\chi))$ are the same.

In the general case of Wolf's theorem, where no monomiality assumption is made, the equality of the numbers $|\text{Irr}^\pi(G)|$ and $|\text{Irr}^\pi(N)|$ is established without constructing a specific bijection. Indeed, in general, no well-behaved bijection exists. If $G = \text{GL}(2, 3)$ and $\pi = \{3\}$, for instance, then all members of $\text{Irr}^\pi(N)$ are rational valued, and yet $\text{Irr}^\pi(G)$ contains irrational characters. (It should be noted, however, that under suitable oddness conditions, but without assuming monomiality, one can get a canonical bijection. The case of this where $|\pi| = 1$ appears as [1, Theorem 10.9].)

The key to the proof of Theorem C, and so to all of the results of this paper is the following.

Lemma D. *Let $K \subseteq G$ and suppose λ is a linear character of K such that λ^G is irreducible. If $S \subseteq K$ is pronormal (i.e. S and S^k are conjugate in $\langle S, S^k \rangle$, for all $k \in K$), then $(\lambda_{K \cap N})^N$ is irreducible, where $N = N_G(S)$.*

We will use Lemma D in the case where G is solvable and S is a Hall π -subgroup of K . Note that if G is arbitrary and S is a Sylow subgroup of K , then S is pronormal in K and the lemma would apply. In this form, with S a Sylow p -subgroup, Lemma D was first proved by G. R. Robinson [4] using module-theoretic methods in characteristic p . (Actually, for Robinson's result, the condition that λ is linear is unnecessarily strong. He needed only that $\lambda(1) \leq (p-1)/2$.) I am doubly indebted to Robinson: First, for challenging me to find a character theoretic proof of his result, and second, for suggesting, over

a year later, that this result (which I had forgotten) might be useful for solving the problem about M -groups on which I was then working.

The M -group result, Theorem A, is a generalization of, and was motivated by, a theorem of T. Okuyama [3] which asserts that N/H is an M -group when H is a Sylow p -subgroup of an M -group G and $N = N_G(H)$. Okuyama's proof depends on characteristic p module theory, and so does not readily generalize to the case where H is a Hall π -subgroup.

2. PROOFS

We begin with a lemma which is an immediate consequence of Mackey's theorem. If $K \subseteq G$ and α is a character of K , then for $g \in G$, we write α^g to denote the character of K^g defined by $\alpha^g(k^g) = \alpha(k)$, for $k \in K$.

2.1. **Lemma.** *Let $K \subseteq G$ and suppose $\alpha \in \text{Irr}(K)$. Then*

(a) α^G is irreducible iff for all $g \in G - K$, we have

$$[\alpha_{K \cap K^g}^g, \alpha_{K \cap K^g}] = 0.$$

(b) If $\beta \in \text{Irr}(L)$ with $L \subseteq G$ and α^G and β^G are both irreducible, then $\alpha^G = \beta^G$ iff

$$[\beta_{K \cap L^g}^g, \alpha_{K \cap L^g}] \neq 0$$

for some $g \in G$.

Proof. We have

$$[\alpha^G, \alpha^G] = [(\alpha^G)_K, \alpha] = \left[\sum_{t \in T} (\alpha'_{K \cap K'})^K, \alpha \right] = \sum_{t \in T} [\alpha'_{K \cap K'}, \alpha_{K \cap K'}],$$

where T is a set of representatives for the double cosets HgH in G . Thus

$$[\alpha^G, \alpha^G] = 1 + \sum_{t \in T_0} [\alpha'_{K \cap K'}, \alpha_{K \cap K'}],$$

where $T_0 = T - K$. Therefore, α^G is irreducible iff

$$[\alpha'_{K \cap K'}, \alpha_{K \cap K'}] = 0$$

for all $t \in T_0$. Since we are free to choose double coset representatives as we will, any element $g \in G - K$ can appear in T_0 , and part (a) follows.

Similarly, using Mackey's theorem, we have

$$[\beta^G, \alpha^G] = [(\beta^G)_K, \alpha] = \sum_{s \in S} [\beta_{K \cap L^s}^s, \alpha_{K \cap L^s}]$$

where S is any set of representatives for the double cosets LgK . Thus $\beta^G = \alpha^G$ iff

$$[\beta_{K \cap L^s}^s, \alpha_{K \cap L^s}] \neq 0$$

for some $s \in S$. Since any element of G can be in S , part (b) is complete. ■

2.2. Lemma. Let $S \subseteq G$ be a pronormal subgroup and suppose that λ, μ are linear characters of G . Then $\lambda = \mu$ iff $\lambda_N = \mu_N$, where $N = \mathbf{N}_G(S)$.

Proof. Since “only if” is trivial, assume $\lambda_N = \mu_N$ and let $M = \ker(\lambda\bar{\mu})$. We must show that $M = G$.

Now $N \subseteq M \triangleleft G$ and the “Fratini argument” proves the result. ■

Our next result contains Lemma D of the Introduction.

2.3. Lemma. Let $K \subseteq G$ and suppose $\lambda^G = \chi \in \text{Irr}(G)$, where λ is a linear character of K . Let S be a pronormal subgroup of K and write $N = \mathbf{N}_G(S)$. Then

(a) $(\lambda_{K \cap N})^N$ is irreducible.

Suppose all that $\mu^G = \psi \in \text{Irr}(G)$ where μ is a linear character of some subgroup L which contains S pronormally.

(b) If $(\lambda_{K \cap N})^N = (\mu_{L \cap N})^N$, then $\chi = \psi$.

Proof. Let $U = K \cap N$. By 2.1 (a), it will follow that $(\lambda_U)^N$ is irreducible if we can show that

$$[\lambda_{U \cap U^n}^n, \lambda_{U \cap U^n}] = 0$$

for every element $n \in N - U$. In other words, we need

$$\lambda_{U \cap U^n} \neq \lambda_{U \cap U^n}^n.$$

Now

$$U \cap U^n = N \cap (K \cap K^n) = \mathbf{N}_{K \cap K^n}(S)$$

and so by 2.2, it suffices to show that

$$\lambda_{K \cap K^n} \neq \lambda_{K \cap K^n}^n.$$

This follows, however, by 2.1 (a), since λ^G is irreducible and $n \notin K$ (because $n \notin U$). This proves (a).

In this situation of (b), write $V = L \cap N$. Since $(\lambda_U)^N = (\mu_V)^N$, Lemma 2.1 (b) gives

$$[\mu_{U \cap V^n}^n, \lambda_{U \cap V^n}] \neq 0$$

for some $n \in N$, and so

$$\mu_{U \cap V^n}^n = \lambda_{U \cap V^n}.$$

Since

$$U \cap V^n = N \cap (K \cap L^n) = \mathbf{N}_{K \cap L^n}(S),$$

we deduce from 2.2 that

$$\mu_{K \cap L^n}^n = \lambda_{K \cap L^n}$$

and thus $\psi = \chi$ by 2.1 (b). ■

Proof of Theorem C. Let $\chi \in \text{Irr}^\pi(G)$. Then χ is monomial and we can write $\chi = \lambda^G$ for some linear character of a subgroup $K \subseteq G$. Also, $|G:K| = \chi(1)$ is a π' -number and so, since G is solvable, K must contain some G -conjugate

of the Hall π -subgroup H . Replacing K by a conjugate, we may assume that $K \supseteq H$.

Since the Hall subgroup H is pronormal in K , Lemma 2.3 applies and we conclude that $\varphi = (\lambda_{K \cap N})^N$ is irreducible. Since $\varphi(1) = |N: K \cap N|$ which divides $|N: H|$, we have $\varphi \in \text{Irr}^\pi(N)$. We shall say in this situation, that φ arises from χ , but we do not yet claim that φ is the only element of $\text{Irr}^\pi(N)$ which arises from χ . By 2.3 (b), however, we know that φ cannot arise from any element $\psi \in \text{Irr}^\pi(G)$ with $\psi \neq \chi$.

If $\mathcal{A} \subseteq \text{Irr}^\pi(N)$ is the set of characters which arise from some $\chi \in \text{Irr}^\pi(G)$, then we have a surjective map $\alpha: \mathcal{A} \rightarrow \text{Irr}^\pi(G)$, defined by $\varphi \mapsto \chi$ if φ arises from χ . Thus $|\mathcal{A}| \geq |\text{Irr}^\pi(G)|$.

Since $|\text{Irr}^\pi(G)| = |\text{Irr}^\pi(N)|$, by the result of Wolf refereed to in the introduction, it follows that $\mathcal{A} = \text{Irr}^\pi(N)$ and that α is injective. The bijection β which we seek is just the inverse of α . ■

Theorem B is now immediate and to prove Theorem A, it suffices to observe that if $N = N_G(H)$, where H is a Hall π -subgroup of G , and $\varphi \in \text{Irr}(N)$, then φ has π' -degree iff φ_H has linear constituents. This occurs iff the derived subgroup $H' \subseteq \ker \varphi$.

ACKNOWLEDGMENT

I would like to thank the Mathematics Research Section of the School of Mathematical Sciences of the Australian National University for their hospitality while this paper was being written.

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