GROWTH FUNCTIONS FOR RESIDUALLY TORSION-FREE NILPOTENT GROUPS

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ABSTRACT. We show that G. Baumslag's growth function for the free group is not algebraic.

Recently, G. Baumslag has introduced some new numerical invariants for the class $\mathcal{T}$ of finitely generated residually torsion-free nilpotent groups. Recall that for $n \geq 1$, we let $\gamma_{n+1}(X) = [\gamma_n(X), X]$, $\gamma_1(X) = X$ and this defines the lower central series of the group X . Since the group $X/\gamma_{n+1}(X)$ is nilpotent it has a torsion subgroup consisting of all the elements of finite order; hence we can define the subgroup $\bar{\gamma}_n(X)$ of X so that $\bar{\gamma}_n(X)/\gamma_{n+1}(X)$ is the torsion subgroup of the quotient group $\gamma_n(X)/\gamma_{n+1}(X)$. The class $\mathcal{T}$ can then be defined to consist of those groups which satisfy

$$\bigcap_n \bar{\gamma}_n(X) = \{1\}.$$

It follows easily from the definition of $\bar{\gamma}_n(X)$ and the finite generation of X that groups in this class have the property that the successive quotients $\bar{\gamma}_n(X)/\bar{\gamma}_{n+1}(X)$ are finitely generated free Abelian groups; let the ranks of these finitely generated free Abelian groups be denoted $r_n(X)$. Let $s_n = \sum_{k=1}^n r_k$; the lower central series dimension of X is

$$\delta(X) = \limsup \left(\frac{\ln(s_n)}{\ln(n)} \right)$$

and their growth series of X is the generating function

$$\text{Gr}(X) = \sum_{n=1}^{\infty} s_n(X) z^n.$$

These invariants of X are analogs of the notion of the dimension of a variety and the Poincaré series of its associated graded local algebra. One would expect then that the size of $\delta(X)$, a measure of the degree of growth of the coefficients $s_n(X)$, is related to the nature of the function $\text{Gr}(X)$, i.e. whether

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it is a polynomial, rational, algebraic or transcendental function. Baumslag has shown that (a) if $X \in \mathcal{F}$ is metabelian then s_n is a polynomial in n for all large n and that δ is an integer and (b) if $X \in \mathcal{F}$ is Abelian by polycyclic then $\text{Gr}(X)$ is a rational function [B]. He has asked if the growth series of the free group of rank q is rational or not. We shall show that it is not algebraic.

The ranks of the quotients of the lower central series of a free group of rank q , $\mathcal{F}_q$, are well known [S, Chapter 4].

$$r_n = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d},$$

where $\mu(d)$ is the Moebius function. Rather than consider the series Gr , we consider the series

$$\text{gr}(z) = \sum_{n=1}^{\infty} r_n(q) z^n.$$

Interchanging the order of summation we obtain

$$\begin{aligned} \text{gr}(z) &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d} z^n \\ &= \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} \frac{\mu(d)}{dm} q^m z^{dm} \\ &= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{m=1}^{\infty} \frac{(qz^d)^m}{m} \\ &= \sum_{d=1}^{\infty} \frac{-\mu(d)}{d} \ln(1 - qz^d). \end{aligned}$$

Formally, we can also obtain the derivative,

$$\text{gr}'(z) = \sum_{d=1}^{\infty} \frac{\mu(d) q z^{d-1}}{1 - qz^d}.$$

Consider this function $\text{gr}(z)$ as a function of the complex parameter z . We observe the following.

Lemma. *If $w(z)$ is an algebraic function then it has at most a finite number of poles; its derivative $w'(z)$ is also an algebraic function.*

Proof. If w satisfies a monic algebraic equation with coefficients which are rational functions then a solution can be found for all values of z outside the finite set of poles of those coefficients; differentiation of the algebraic equation satisfied by w gives an algebraic expression for w' in terms of w and the conclusion follows.

Thus to show that $\text{gr}(z)$ is not algebraic it suffices to show that its derivative has an infinite number of poles. Of course this is in some sense obvious since the poles occur at $q^{-1/d}$ for square-free values of d . However we need to

verify that this is in fact a valid series for $\text{gr}'(z)$ and that the poles do not somehow cancel out.

Proposition. *The function $\text{gr}'(z)$ is not algebraic and hence neither is $\text{gr}(z)$.*

Proof. An easy comparison with the geometric series shows that the function $\text{gr}'(z)$ converges uniformly for $|z| < 1/q$ and hence that $\text{gr}(z)$ converges uniformly there also. Moreover, if we consider $\text{gr}'(z)$ on any compact subset of $|z| < 1$ it is easily seen to be meromorphic there since we need only delete a finite number of terms from its series expression above to get a holomorphic series. Thus we shall show that $\text{gr}(z)$ is not algebraic by showing that $\text{gr}'(z)$ has simple poles at the values $q^{-1/d}$ for square-free values of d , i.e., an infinite number of poles. Since the series for $\text{gr}'(z)$ converges absolutely in a deleted neighborhood of some particular $z_0 = q^{-1/d}$ we evaluate:

$$\lim_{z \rightarrow z_0} (z - q^{1/d}) \text{gr}'(z) = \lim_{z \rightarrow z_0} (z - q^{-1/d}) \frac{\mu(d)qz^{d-1}}{1 - qz^d} \neq 0$$

and the conclusion now follows.

REFERENCES

- [B] G. Baumslag, *Some reflections on finitely generated metabelian groups*, Contemp. Math. (to appear).
- [S] J.-P. Serre, *Lie algebras and Lie groups*, Benjamin, New York, 1965.

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