AN EXTREME ABSOLUTELY CONTINUOUS RP -MEASURE

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ABSTRACT. We construct a measure on the torus T^2 which is an extreme element of the set of RP -probability measures and is absolutely continuous with respect to Haar measure on T^2 .

The well-known theorem of Herglotz asserts that, if u is a positive harmonic function on the open unit disk D , then there is a unique Borel measure μ on the unit circle T such that

$$(1) \quad u(z) = \int_T P_z(x) d\mu(x),$$

where $P_z(x)$ denotes the Poisson kernel $\operatorname{Re}(x+z)/(x-z)$. An equivalent way to state Herglotz's theorem is the following: Let f be holomorphic in D , have positive real part, and satisfy $f(0) = 1$. Then there is a unique probability measure μ_f on T such that

$$(2) \quad \operatorname{Re} f(z) = \int_T P_z(x) d\mu_f(x).$$

It is clear from (2) that the correspondence $M_1(f) = \mu_f$ is a bijection between the convex sets $P(T) =$ Borel probability measures on T and

$$\mathcal{P}_1 = \{f | f \text{ holomorphic in } D, \operatorname{Re} f > 0, f(0) = 1\}.$$

Also, if $\mathcal{P}_1$ and $P(T)$ are equipped with the topology of uniform convergence on compacta and the weak* topology, respectively, then M_1 is continuous. Moreover, M_1 is affine, i.e. it preserves convex combinations. It follows that

$$M_1(\operatorname{ex} \mathcal{P}_1) = \operatorname{ex} P(T),$$

where $\operatorname{ex} \mathcal{P}_1$ denotes the set of extreme elements of the set $\mathcal{P}_1$. Since the extreme elements of the set $P(T)$ are exactly the measures supported by singletons, it follows easily that a function f in $\mathcal{P}_1$ is extreme iff

$$f(z) = (y+z)/(y-z) \quad \text{for some } y \text{ in } T.$$

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When the disk D is replaced by the n -dimensional unit polydisk D^n , there is an analogue of (1). Namely, if $u(z) = u(z_1, z_2, \dots, z_n)$ is positive on D^n and harmonic in each variable, then there is a unique Borel measure μ on the torus T^2 with

$$u(z) = \int_{T^n} P_{z_1}(x_1) P_{z_2}(x_2) \cdots P_{z_n}(x_n) d\mu(x).$$

Thus, as in the one-dimensional case, we have a mapping $M_n: \mathcal{P}_n \rightarrow P(T^n)$, where

$$\mathcal{P}_n = \left\{ f \mid f \text{ holomorphic in } D^n, \operatorname{Re} f > 0, f(0, \dots, 0) = 1 \right\}$$

and $M_n(f)$ is the unique probability measure μ_f satisfying

$$\operatorname{Re} f(z) = \int_{T^n} P_{z_1}(x_1) P_{z_2}(x_2) \cdots P_{z_n}(x_n) d\mu_f(x).$$

Like M_1 , the mapping M_n is affine, continuous, and one-to-one, but, for $n > 1$, it is not onto. (It is not hard to show that a probability measure μ belongs to $M_2(\mathcal{P}_2)$ iff it satisfies: $\int_{T^2} x^p y^q d\mu(x, y) = 0$ for all pairs of integers (p, q) with $pq < 0$.) The set $M_n(\mathcal{P}_n)$, which will be denoted by RP_n , is weak* compact and convex and, hence, it must have extreme points. But, unlike the situation in the case $n = 1$, the problem of describing the extreme elements of RP_n for $n > 1$ remains open, although some examples and partial results have been obtained [1, 3–5]. (Note that, for $n > 1$, RP_n contains no point masses.) In [6] Rudin asked the question of whether or not there can be extreme elements of RP_n which are absolutely continuous with respect to Haar measure on T^n . In this paper we answer Rudin's question affirmatively.

MAIN RESULT

From now on we will work with the case $n = 2$. We will write RP for RP_2 and $\mathcal{P}$ for $\mathcal{P}_2$. The closed unit ball in the space of bounded holomorphic functions on D will be denoted by B . Our key result is the following:

Theorem 1. *Let $g \in \operatorname{ex} B$. Suppose that $g(0) = 0$. Then the function*

$$(3) \quad F(z, w) = \frac{(1-z)(1-g(w))}{1-zg(w)}$$

is an extreme element of $\mathcal{P}$.

(This result was proved in [4] in the case where g is an inner function.)

It can be shown that there is a function g in B which is extreme, vanishes at 0, and satisfies $|g(e^{it})| = \lim_{r \rightarrow 1} |g(re^{it})| < 1$ for almost all $t \in [0, 2\pi)$. See [2, p. 138]. For such g we construct a measure ν on T^2 via

$$d\nu(x, y) = (2\pi)^{-2} \operatorname{Re} \frac{(1-x)(1-g(y))}{1-xg(y)} |dx| |dy|.$$

It is easy to show that the Poisson integral of ν is

$$\operatorname{Re} \frac{(1-z)(1-g(w))}{1-zg(w)}.$$

It follows that ν is an absolutely continuous extreme element of RP .

The remainder of the paper will be devoted to the proof of Theorem 1. We will need the following well-known result:

Theorem 2 (Caratheodory). *Let $f \in \mathcal{P}_1$ and let the Taylor series expansion of f about $z = 0$ be written*

$$f(z) = 1 + 2 \sum_{n=1}^{\infty} a_n z^n.$$

Then $|a_n| \leq 1$ for $n = 1, 2, \dots$

Suppose now that $G \in \mathcal{P}$. We expand $G(z, w)$ in a series of the form

$$G(z, w) = g_0(w) + 2 \sum_{n=1}^{\infty} g_n(w) z^n.$$

Applying Theorem 2 to the function

$$f(z) = (G(z, w) - i \operatorname{Im} g_0(w)) / \operatorname{Re} g_0(w)$$

we obtain the following:

$$(4) \quad |g_n(w)| \leq \operatorname{Re} g_0(w), \quad n = 1, 2, \dots, w \in D.$$

We are now ready to begin the proof of Theorem 1. Let F be as in the statement of the theorem. In order to show that F is an extreme element of $\mathcal{P}$ it suffices to show that any function U satisfying: $F \pm U \in \mathcal{P}$ must vanish identically. We define holomorphic functions $u_0, u_1, u_2, \dots$ via the expansion

$$U(z, w) = u_0(w) + 2 \sum_{n=1}^{\infty} u_n(w) z^n.$$

Applying (4) to the functions $F \pm U$, we obtain

$$(5) \quad \left| g^n - (g^{n-1} + g^{n+1}) / 2 \pm u_n \right| \leq 1 - \operatorname{Re} g \pm \operatorname{Re} u_0.$$

From (5) the inequalities

$$(6) \quad |u_n| \leq 2, \quad n = 1, 2, \dots$$

follow via

$$\begin{aligned} 2|u_n| &\leq \left| g^n - (g^{n-1} + g^{n+1}) / 2 + u_n \right| + \left| g^n - (g^{n-1} + g^{n+1}) / 2 - u_n \right| \\ &\leq 1 - \operatorname{Re} g + \operatorname{Re} u_0 + 1 - \operatorname{Re} g - \operatorname{Re} u_0 \end{aligned}$$

and $|g| \leq 1$.

We will derive more inequalities involving the functions $\{u_n\}$ from the following:

$$(7) \quad \begin{aligned} 0 < 1 - |z|^2 |g(w)|^2 - (1 - |g(w)|^2) \operatorname{Re} z \\ - (1 - |z|^2) \operatorname{Re} g(w) \pm \operatorname{Re} |1 - zg(w)|^2 U(z, w), \end{aligned}$$

for $z, w \in D$. The inequality (7) follows at once from

$$0 < \operatorname{Re} (F(z, w) \pm U(z, w)).$$

Let $\alpha(z, w)$ denote the right-hand side of (7) and let

$$\beta_k(r, w) = (2\pi)^{-1} \int_0^{2\pi} \alpha(re^{i\theta}, w) e^{-ik\theta} d\theta,$$

where $0 \leq r < 1$ and $k = 0, 1, 2, \dots$. We have immediately

$$(8) \quad 0 \leq \beta_0(r, w),$$

$$(9) \quad |\beta_k(r, w)| \leq \beta_0(r, w).$$

Straightforward calculations show that

$$\begin{aligned} \beta_0(r, w) &= 1 - r^2 |g(w)|^2 - (1 - r^2) \operatorname{Re} g(w) \\ &\quad \pm \left((1 + r^2 |g(w)|^2) \operatorname{Re} u_0(w) - 2r^2 \operatorname{Re} (\overline{g(w)} u_1(w)) \right), \\ \beta_1(r, w) &= -r (1 - |g(w)|^2) / 2 \pm r \left((1 + r^2 |g(w)|^2) u_1(w) \right. \\ &\quad \left. - rg(w) \operatorname{Re} u_0(w) - r^3 u_2(w) \overline{g(w)} \right), \\ \beta_k(r, w) &= \pm \left((1 + r^2 |g(w)|^2) r^k u_k(w) - r^k u_{k-1}(w) g(w) \right. \\ &\quad \left. - r^{k+2} u_{k+1}(w) \overline{g(w)} \right), \quad \text{for } k \geq 2. \end{aligned}$$

Letting r approach 1 in (8) and (9), we obtain

$$(10) \quad 0 \leq 1 - |g|^2 \pm \left((1 + |g|^2) \operatorname{Re} u_0 - 2 \operatorname{Re} \overline{g} u_1 \right),$$

$$(11) \quad \begin{aligned} \left| - (1 - |g|^2) / 2 \pm \left((1 + |g|^2) u_1 - g \operatorname{Re} u_0 - u_2 \overline{g} \right) \right| \\ \leq 1 - |g|^2 \pm \left((1 + |g|^2) \operatorname{Re} u_0 - 2 \operatorname{Re} \overline{g} u_1 \right), \end{aligned}$$

$$(12) \quad \begin{aligned} \left| (1 + |g|^2) u_k - u_{k-1} g - u_{k+1} \overline{g} \right| \\ \leq 1 - |g|^2 \pm \left((1 + |g|^2) \operatorname{Re} u_0 - 2 \operatorname{Re} \overline{g} u_1 \right). \end{aligned}$$

Multiplying both sides of (5) by $|\overline{g}|$ and using $|g| \leq 1$ we obtain

$$\left| -|g|^2 + (g|g|^2 + \overline{g}) / 2 \mp \overline{g} u_1 \right| \leq 1 - \operatorname{Re} g \pm \operatorname{Re} u_0.$$

It follows that

$$|1 - \operatorname{Re} g \pm \bar{g}u_1| \leq 1 - \operatorname{Re} g \pm \operatorname{Re} u_0 + |1 - g/2| (1 - |g|^2).$$

Hence,

$$(13) \quad |1 - \operatorname{Re} g \pm \bar{g}u_1| \leq 1 - \operatorname{Re} g \pm \operatorname{Re} u_0 + 2(1 - |g|^2).$$

Next we use (10) to get

$$\begin{aligned} \mp \operatorname{Re} u_0 &\leq (1 - |g|^2) / (1 + |g|^2) \mp 2 \operatorname{Re} \bar{g}u_1 / (1 + |g|^2) \\ &\leq 1 - |g|^2 \mp \operatorname{Re} \bar{g}u_1 + (1 - |g|^2) |\operatorname{Re} \bar{g}u_1|. \end{aligned}$$

From (6) and the fact that $|g| \leq 1$ it follows easily that

$$(14) \quad \mp \operatorname{Re} u_0 \leq 3(1 - |g|^2) \mp \operatorname{Re} \bar{g}u_1.$$

Combining (13) and (14) we obtain

$$|1 - \operatorname{Re} g \pm \bar{g}u_1| \leq 1 - \operatorname{Re} g \pm \operatorname{Re} \bar{g}u_1 + 5(1 - |g|^2).$$

Hence,

$$\begin{aligned} (1 - \operatorname{Re} g \pm \operatorname{Re} \bar{g}u_1)^2 + (\operatorname{Im} \bar{g}u_1)^2 &\leq (1 - \operatorname{Re} g \pm \operatorname{Re} \bar{g}u_1)^2 \\ &\quad + 10(1 - |g|^2)(1 - \operatorname{Re} g \pm \operatorname{Re} \bar{g}u_1) \\ &\quad + 25(1 - |g|^2)^2. \end{aligned}$$

Using $1 - \operatorname{Re} g \mp \operatorname{Re} \bar{g}u_1 \leq 4$ and $1 - |g|^2 \leq 1$ we get

$$(15) \quad (\operatorname{Im} \bar{g}u_1)^2 \leq 65(1 - |g|^2).$$

Next we prove the inequality

$$(16) \quad |\operatorname{Re} u_0 - \operatorname{Re} (\bar{g}u_1)| \leq \left(\frac{3}{2}\right)(1 - |g|^2).$$

For by (10) we have

$$|2 \operatorname{Re} u_0 - (1 - |g|^2) \operatorname{Re} u_0 - 2 \operatorname{Re} (\bar{g}u_1)| \leq 1 - |g|^2.$$

Thus,

$$2|\operatorname{Re} u_0 - \operatorname{Re} (\bar{g}u_1)| \leq (1 - |g|^2)(1 + |\operatorname{Re} u_0|).$$

By setting $z = 0$ in (7) we obtain

$$(17) \quad |\operatorname{Re} u_0| \leq 2$$

from which (16) follows immediately.

We are now ready to show that

$$(18) \quad u_2 = gu_1.$$

From (10) and (11) we have

$$\left| -\left(1 - |g|^2\right) / 2 \pm \left(\left(1 + |g|^2\right) u_1 - g \operatorname{Re} u_0 - u_2 \bar{g} \right) \right| \leq 2 \left(1 - |g|^2\right).$$

By the triangle inequality and (6) we obtain

$$\left| 2u_1 - g \operatorname{Re} u_0 - u_2 \bar{g} \right| \leq \frac{9}{2} \left(1 - |g|^2\right).$$

Multiplying both sides of the last inequality by $|\bar{g}|$ and again applying the triangle inequality we get

$$\begin{aligned} \left| 2\bar{g}u_1 - \operatorname{Re} u_0 - u_2 \bar{g}^2 \right| &\leq \left(\frac{9}{2} + |\operatorname{Re} u_0| \right) \left(1 - |g|^2\right) \\ &\leq \frac{13}{2} \left(1 - |g|^2\right). \end{aligned}$$

It follows that

$$\left| \bar{g}u_1 - \bar{g}^2u_2 \right| \leq \frac{13}{2} \left(1 - |g|^2\right) + |\operatorname{Re} u_0 - \operatorname{Re}(\bar{g}u_1)| + |\operatorname{Im} \bar{g}u_1|.$$

Hence, by (15) and (16) we have

$$\begin{aligned} \left| \bar{g}u_1 - \bar{g}^2u_2 \right| &\leq 8 \left(1 - |g|^2\right) + \left(65 \left(1 - |g|^2\right) \right)^{1/2} \\ &\leq 17 \left(1 - |g|^2\right)^{1/2}. \end{aligned}$$

Thus, using (6) we have

$$\begin{aligned} \left| u_1 - \bar{g}u_2 \right| &= |g| \left| u_1 - u_2 \bar{g} \right| + (1 - |g|) \left| u_1 - u_2 \bar{g} \right| \\ &\leq 17 \left(1 - |g|^2\right)^{1/2} + 4(1 - |g|). \\ &\leq 38(1 - |g|)^{1/2}. \end{aligned}$$

It now follows easily that

$$\begin{aligned} \left| gu_1 - u_2 \right| &\leq 38(1 - |g|)^{1/2} + 2 \left(1 - |g|^2\right) \\ &\leq 42(1 - |g|)^{1/2}. \end{aligned}$$

Thus,

$$|g| + \left| ((gu_1 - u_2) / 42)^2 \right| \leq 1.$$

Since g is an extreme point of B , it follows that (18) is valid.

Next we will show that

$$(19) \quad u_{k+1} = 2gu_k - g^2u_{k-1}, \quad \text{for } k = 2, 3, \dots$$

From (10) and (12) we have

$$\left| \left(1 + |g|^2\right) u_k - gu_{k-1} - \bar{g}u_{k+1} \right| \leq 2 \left(1 - |g|^2\right).$$

By application of (6) and the triangle inequality we get

$$\left| 2gu_k - g^2u_{k-1} - u_{k+1} \right| \leq 6 \left(1 - |g|^2\right).$$

It follows that

$$\left| (2gu_k - g^2u_{k-1} - u_{k+1}) / 12 \right| + |g| \leq 1.$$

Hence, (19) is valid because of the extremality of g .

Using (18) and (19) it is easy to prove by induction that $u_k = g^{k-1}u_1$ for $k = 1, 2, \dots$. It follows that the inequality (7) may be rewritten in the form

$$(20) \quad 0 < 1 - |z|^2 |g|^2 - (1 - |g|^2) \operatorname{Re} z - (1 - |z|^2) \operatorname{Re} g \\ \pm \operatorname{Re} |1 - zg|^2 (u_0 + 2u_1 z / (1 - zg)).$$

Letting $z = 1$, we obtain

$$0 \leq \pm \operatorname{Re} (u_0 + 2u_1 / (1 - g)).$$

It follows that

$$u_0 + 2u_1 / (1 - g) = ic,$$

where c is a real constant. Since $g(0) = u_0(0) = 0$, we must have $2u_1(0) = ic$. Returning to (20) and letting $w = 0$, we get

$$0 \leq 1 - \operatorname{Re} z \pm \operatorname{Re} icz$$

for all z with $|z| \leq 1$. By choosing $z = (1 \pm ic)/(1 + c^2)^{1/2}$ we obtain $1 \geq (1 + c^2)^{1/2}$. Thus, $c = 0$.

We now rewrite (20) in the form

$$0 < 1 - |z|^2 |g|^2 - (1 - |g|^2) \operatorname{Re} z - (1 - |z|^2) \operatorname{Re} g \\ \pm \operatorname{Re} 2|1 - zg|^2 u_1 (z - 1) / (1 - zg)(1 - g).$$

Thus,

$$0 < |1 - g|^2 (1 - |z|^2 |g|^2 - (1 - |g|^2) \operatorname{Re} z - (1 - |z|^2) \operatorname{Re} g) \\ \pm \operatorname{Re} 2u_1 (z - 1)(1 - \overline{zg})(1 - \overline{g}).$$

Passing to the case $|z| = 1$ and using the inequality $|1 - g| \leq 2$, we obtain

$$0 \leq 2(1 - |g|^2)(1 - \operatorname{Re} z) \\ \pm \operatorname{Re} u_1 (z - 1 + (\overline{z} - z)\overline{g} + (1 - \overline{z})\overline{g}^2).$$

Thus,

$$(21) \quad 0 \leq A - Ax + By,$$

where $z = x + iy$,

$$A = 2(1 - |g|^2) \pm \operatorname{Re} u_1 (\overline{g}^2 - 1),$$

and

$$B = \pm \operatorname{Re} iu_1 (1 - 2\overline{g} + \overline{g}^2).$$

By minimizing the right-hand side of (21) we obtain

$$(A^2 + B^2)^{1/2} \leq A.$$

It follows that $B = 0$. Hence, we have

$$\operatorname{Re} iu_1 / (1 - g)^2 = 0.$$

Thus, $iu_1 / (1 - g)^2$ is constant. But $u_1(0) = 0$. Hence u_1 must vanish identically. Since we have shown that u_k vanishes identically for $k = 0, 1, 2, \dots$, the proof of Theorem 1 is complete.

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