UNIQUE SUBWORDS IN NONPERIODIC WORDS

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ABSTRACT. Let A, D be words over some alphabet. D has position p in the cyclic word A if the cyclic permutation of A which begins with the p th letter of A has an initial subword equal to D . It is proved that every nonperiodic word A of length > 1 has a cyclic permutation which is a product BC for some nonempty subwords B, C having unique positions in the cyclic word A .

1. INTRODUCTION

The result in the Abstract can be viewed as an assertion about a group $\langle a, b, \dots; A = 1 \rangle$ with a single nonperiodic defining relator A involving 2 or more generators $a, b, \dots$ but none of the inverses of $a, b, \dots$. The result may be useful for investigating the group's Cayley graph. (See [1, p. 57] for a definition.) We conjecture that a general result holds for any presentation $\langle a, b, \dots; A = 1 \rangle$ whenever A is nonperiodic with length > 1 .

Conjecture. Every nonperiodic word A of length > 1 has a cyclic permutation which is a product BC where nonempty words B, C have unique positions in the cyclic word A and B, C have no positions in the cyclic word formed by the inverse of A .

For convenient exposition, finite sequences instead of words will be used to introduce terminology and state and prove the main result in Theorem 1.

2. TERMINOLOGY FOR SEQUENCES

A finite nonempty sequence of objects is referred to briefly as a *sequence*. A well-defined relation of equality between objects is assumed. The length n of a sequence S is denoted by $|S|$. A sequence T is an *initial subsequence* of S if T consists of the first $|T|$ terms of S . An *extreme term* of S is either the first or last term of S . There is just one extreme term if $|S| = 1$. $S(j)$ denotes the

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j th term of S . $[1, n]$ denotes the set of allowable values of j , namely, the set of integers from 1 to n , inclusive. An integer p in $[1, n]$ is a *position* for S .

Let p be a position for a sequence S . Let (Sp) denote the cyclic permutation of S whose first term is the p th term of S . Then a sequence T is a *segment* of S , having position p , if T is an initial subsequence of (Sp) . The *product* UV of sequences U, V is a sequence consisting of the terms of U followed by the terms of V . The finite product P , formed from a sequence S of sequences, is defined similarly. S is referred to as a *factorization* of P ; each term in S is a *factor*.

A sequence S is *periodic* if, for some integer $k > 1$, S is the product of k copies of some sequence X . The *period* of S , denoted by $\underline{S}$, is equal to $|X|$ when k is as large as possible.

3. TERMINOLOGY FOR RUN SEQUENCES

Choose an object z which may or may not be equal to some term in a given sequence A . The *run sequence* rA , based on z , is the factorization of A , with as few factors as possible, such that, for each factor, either all or none of the terms in that factor are equal to z . A *run* is a term in rA . If p is a position for A and $A(p)$ is the first term in some run, then p is a *starting position* for A . The sequence of starting positions for A is denoted by sA . The sequences of odd- and even-numbered terms of rA are denoted by orA, erA , respectively. The expressions $sA(j), rA(j), orA(j), erA(j)$ denote the j th terms of sA, rA, orA, erA , respectively.

4. PRELIMINARIES

Lemma 1. Let sequence B be a segment of a sequence A . Assume unequal extreme terms for A . Let rB, rA be the run sequences based on $A(1)$ so length of rA is $2m$ for some positive integer m . Denote the sequences of odd-numbered terms of rB, rA by orB, orA , respectively. Assume

- (1) each extreme term of B is equal to $A(1)$,
- (2) each extreme term of rB is a sequence of length $oMAX$ where $oMAX$ is the maximum of the lengths of the sequences which appear as terms in orA ,
- (3) orB has a unique position in orA .

Then B has a unique position in A .

Proof. Suppose B has position b in A . Conditions (1), (2) imply that the extreme terms of rB are runs in rA . Hence, rB is a segment of rA with odd position $2k - 1$ in rA for some integer k in $[1, m]$. Thus orB has position k in orA . But k is unique by (3). Hence rB, B have unique positions $2k - 1, sA(2k - 1)$ in rA, A respectively. (See §3 for definition of sA .) So $b = sA(2k - 1)$ and b is unique.

Lemma 2. Let sequence C be a segment of a sequence A . Assume unequal extreme terms for A . Assume C has a factorization given by $C = XWY$. Let rW, rA be the run sequences based on $A(1)$ so length of rA is $2m$ for some

positive integer m . Denote the sequences of odd-numbered terms of rW , rA by orW , orA , respectively. Assume that

- (1) sequences X, Y contain no terms equal to $A(1)$,
- (2) each extreme term of W is equal to $A(1)$,
- (3) orW has a unique position in orA .

Then C has a unique position in A .

Proof. Suppose C has position c in A . Conditions (1), (2) imply rW is a segment of rA . So rW has odd position $2k-1$ in rA for some integer k in $[1, m]$. Thus orW has position k in orA . But k is unique by (3). Hence, rW, W have unique positions $2k-1, sA(2k-1)$ in rA, A , respectively. (See §3 for definition of sA .) So c is congruent, modulo $|A|$, to $sA(2k-1) - |X|$. Hence, c is unique.

Lemma 3. Let a sequence U have positions c, d in a periodic sequence V with period $\underline{V}$. If $|U| = \underline{V} - 1$, then c, d are congruent modulo $\underline{V}$.

Lemma 4. Let S, T be sequences with S a segment (of T) having an odd position in T . Assume

- (1) T has even length ($|T| = 2m$ with m a positive integer),
- (2) the odd and even numbered subsequences oT, eT of T have no common terms,
- (3) oT, eT are periodic with periods $\underline{oT}, \underline{eT}$, satisfying $1 < \underline{eT} < \underline{oT}$,
- (4) m is the least common multiple of $\underline{oT}$ and $\underline{eT}$,
- (5) $|S| = 2\underline{oT} - 3$.

Then S has a unique position in T .

Proof. Suppose S has positions p, q in T . Then oS has positions $(1+p)/2, (1+q)/2$ in oT . Also eS has positions $(1+p)/2, (1+q)/2$ in eT and so does eW for each initial subsequence W of S . Define such a W by requiring $|W| = 2\underline{eT} - 2$. Then $|W|$ does not exceed $|S|$ because of (3), (5). Note that $|eW| = \underline{eT} - 1$. Apply Lemma 3 to $V = oT, U = oS$ and to $V = eT, U = eW$ to find $(1+p)/2, (1+q)/2$ congruent modulo $\underline{oT}$, modulo $\underline{eT}$ and hence modulo m , using (4). Hence, $1+p, 1+q$ are congruent modulo $2m$, i.e. modulo $|T|$. So p, q are positions for T and are congruent modulo $|T|$. Thus $p = q$.

5. MAIN RESULT

Theorem 1. Each finite nonperiodic sequence A of length $n > 1$ has a cyclic permutation which can be expressed as a product BC where each of the sequences B, C has a unique position in A . For any integer p in $[1, n]$, the sequences B, C can be chosen so that each extreme term of B is equal to $A(p)$ and each extreme term of C is not equal to $A(p)$.

Proof. Use induction on n . The theorem is obvious when $n = 2$. Unequal terms $A(1), A(n)$ and $p = 1$ can be assumed. Each run sequence defined in the proof, including rA , is based on the object $A(1)$. Denote the odd-

and even-numbered subsequences of rA by orA , erA , respectively. Then $|rA| = 2m$, $|orA| = |erA| = m$ for some positive integer m . In each of five cases, definitions of B , C assure that the extreme terms of B , C satisfy the theorem.

Case 1. $m = 1$. In this case, use $B = rA(1)$ and $C = rA(2)$.

Case 2. orA is not periodic and $m > 1$.

Define $oMAX$ to be the maximum of $|orA(j)|$ for j in $[1, m]$. Pick an integer q in $[1, m]$ so that $|orA(q)| = oMAX$. By the induction assumption we can apply Theorem 1 to the sequence orA . Thus there exist sequences P , Q such that PQ is a cyclic permutation of orA , the positions for P , Q in orA are unique, and each extreme term of P is equal to $orA(q)$. Now assume that $orA = PQ$ because, if not, the remainder of the proof for Case 2 can be applied to the cyclic permutation A' of A for which $orA' = PQ$. Then the sequences B , C found for A' will suffice for A .

Define B to be the product of the first $2|P| - 1$ terms in rA . Then rB , orB are initial subsequences of rA , orA , respectively and $orB = P$. Apply Lemma 1 to conclude that B has a unique position in A .

Define C by $A = BC$. Then each extreme term of C is not equal to $A(1)$ because each extreme term of rB has length $oMAX$. Thus $rA = rBrC$. So $C = XWY$ where X , Y are the extreme terms of rC and $orW = Q$. Apply Lemma 2 to conclude that C has a unique position in A .

Case 3. erA is not periodic and $m > 1$.

Let $A = XY$ where $X = rA(1)$. Let $D = YX$. Then $orD = erA$. So Case 2 applies to D .

Case 4. orA and erA are periodic (hence, $m > 1$) and the periods $\underline{orA}$, $\underline{erA}$, satisfy $\underline{orA} < \underline{erA}$.

In this case, m equals the least common multiple of $\underline{orA}$ and $\underline{erA}$ else A would be periodic. Define B to be the product of the first $2\underline{erA} - 1$ terms in rA . Define C by $A = BC$. Then $B = XYZ$ where X , Z are the first, last runs of B and Y is the product of the middle $2\underline{erA} - 3$ runs of B . Thus rY is a segment with position 2 in rA , with $|rY| = 2\underline{erA} - 3$. Using $D = YZCX$, we find that rY is an initial subsequence of rD and $\underline{orD} = \underline{erA}$, $\underline{erD} = \underline{orA}$. Apply Lemma 4 using $T = rD$, $S = rY$. So rY has a unique position in rD ; hence rY has a unique position, namely 2, in rA .

Now suppose B has position b in A . Let i , j denote the positions for A which are congruent modulo $|A|$ to $b + |X| - 1$ and $b + |X|$, respectively. Then the last term of X and the first term of Y have positions i , j , respectively, in A . So Y has position j in A . Since all the terms in X and none of the extreme terms in Y are equal to $A(1)$, j is a starting position for A , appearing as an even-numbered term in sA (which is defined in §3). Thus $j = sA(2k)$ for some integer k in $[1, m]$. Therefore rY has position $2k$ in rA , but this position is unique and equal to 2. So $j = sA(2k) = sA(2) = 1 + |X|$. Hence

$1 + |X|$, $b + |X|$ are congruent modulo $|A|$. Thus $b = 1$ and B has a unique position in A .

Now apply the reasoning about B to another segment H (of A), also having an initial subsequence equal to an odd-numbered run of A . Define H to be the product of $2\text{er}A - 1$ terms in rA , beginning with the term $rA(2\text{er}A + 1)$. Deduce that H has a unique position in A . Define C by $A = BC$. Then C has a unique position in A because so does H , a subsequence of C consisting of consecutive terms of C .

Case 5. orA and erA are periodic (hence, $m > 1$) and periods orA , erA , satisfy $erA < orA$.

Let $A = GH$ where G is the first run in rA . Define $A' = HG$ so that the last term of A' equals $A(1)$ and A' has unequal extreme terms. Then $orA' = erA$ and $erA' = orA$. Now apply the reasoning in Case 4 to A' .

Define B' to be the product of the first $2\text{er}A' - 1$ terms in rA' . Define C' by $A' = B'C'$. Each extreme term of B' equals $A'(n)$ and each extreme term of C' equals $A(1)$. As in Case 4, B' , C' have unique positions in A' . Now use $B = C'$, $C = B'$ so that BC is a cyclic permutation of $B'C'$ which is a cyclic permutation of A . This completes the proof.

REFERENCES

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