EVALUATION OF THE INFINITE SERIES $\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1}$

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ABSTRACT. Let p be a prime $\equiv 1 \pmod{4}$. The value of the infinite series $\sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1}$ is given in finite terms.

Let p be a prime $\equiv 1 \pmod{4}$. If n is an integer such that the Legendre symbol $\left(\frac{n}{p}\right)$ has the value $+1$, then the Dirichlet symbol $\left(\frac{n}{p}\right)_4$ of quartic residuacity is defined by

$$\left(\frac{n}{p}\right)_4 = \begin{cases} +1, & \text{if } x^4 \equiv n \pmod{p} \text{ is solvable,} \\ -1, & \text{if } x^4 \equiv n \pmod{p} \text{ is unsolvable.} \end{cases}$$

In this note we use Mathews' evaluation [3] of the quartic Gauss sum to determine the sum of the infinite series

$$(1) \quad \sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1}.$$

We define integers a and b uniquely by

$$(2) \quad \begin{cases} p = a^2 + b^2, \\ a \equiv 1 \pmod{4}, b \equiv -((p-1)/2)!a \pmod{p}. \end{cases}$$

We set

$$(3) \quad \omega = a + ib, \quad (\text{so that } \omega\bar{\omega} = p),$$

and define a quartic character χ_{ω} on the ring of Gaussian integers $Z[i] \bmod \omega$ by

$$\chi_{\omega}(\nu) \equiv \nu^{(p-1)/4} \pmod{\omega}.$$

Thus, for an integer $n \not\equiv 0 \pmod{p}$, we have

$$\chi_{\omega}(n) = \left(\frac{n}{p}\right)_4, \quad \text{if } \left(\frac{n}{p}\right) = 1,$$

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and

$$\frac{\chi_{\omega}(n)}{i} = \pm 1, \quad \text{if } \left(\frac{n}{p}\right) = -1.$$

We also let

$$(4) \quad LS_{\omega}(i^k) = \sum_{\substack{s=1 \\ \chi_{\omega}(s)=i^k}}^{p-1} \log \sin(\pi s/p) \quad (k = 0, 1, 2, 3)$$

and

$$(5) \quad F_{\omega}(i^k) = \sum_{\substack{s=1 \\ \chi_{\omega}(s)=i^k}}^{p-1} s \quad (k = 0, 1, 2, 3).$$

With the above notation, we have

Theorem. (a) For $p \equiv 1 \pmod{8}$ we have

$$(6) \quad \sum_{\substack{n=1 \\ \left(\frac{n}{p}\right)=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} = (2p)^{-1} (-1)^{(p+2b+7)/8} \left(\frac{2|b|}{|a|}\right) \\ \times \left(-\frac{b}{|b|} \sqrt{2p - 2a\sqrt{p}} (LS_{\omega}(1) - LS_{\omega}(-1)) \right. \\ \left. + \sqrt{2p + 2a\sqrt{p}} (LS_{\omega}(i) - LS_{\omega}(-i)) \right),$$

and for $p \equiv 5 \pmod{8}$ we have

$$(7) \quad \sum_{\substack{n=1 \\ \left(\frac{n}{p}\right)=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} = (2p^2)^{-1} (-1)^{(p+2b+7)/8} \left(\frac{2|b|}{|a|}\right) \\ \times \pi \left(\sqrt{2p + 2a\sqrt{p}} (F_{\omega}(1) - F_{\omega}(-1)) \right. \\ \left. + \frac{b}{|b|} \sqrt{2p - 2a\sqrt{p}} (F_{\omega}(i) - F_{\omega}(-i)) \right).$$

(b) For $p \equiv 1 \pmod{8}$ we have

$$(8) \quad \sum_{\substack{n=1 \\ \left(\frac{n}{p}\right)=-1}}^{\infty} \left(\frac{\chi_{\omega}(n)}{i}\right) n^{-1} = (2p)^{-1} (-1)^{(p+2b+7)/8} \left(\frac{2|b|}{|a|}\right) \\ \times \left(\sqrt{2p + 2a\sqrt{p}} (LS_{\omega}(1) - LS_{\omega}(-1)) \right. \\ \left. + \frac{b}{|b|} \sqrt{2p - 2a\sqrt{p}} (LS_{\omega}(i) - LS_{\omega}(-i)) \right),$$

and for $p \equiv 5 \pmod{8}$ we have

$$(9) \quad \sum_{\substack{n=1 \\ \left(\frac{a}{p}\right)=-1}}^{\infty} \left(\frac{\chi_{\omega}(n)}{i}\right) n^{-1} = (2p^2)^{-1} (-1)^{(p+2b+7)/8} \left(\frac{2|b|}{|a|}\right) \\ \times \pi \left(\frac{b}{|b|} \sqrt{2p - 2a\sqrt{p}} (F_{\omega}(1) - F_{\omega}(-1)) \right. \\ \left. - \sqrt{2p + 2a\sqrt{p}} (F_{\omega}(i) - F_{\omega}(-i)) \right).$$

(The large parentheses denote the usual Jacobi symbol.)

We remark that

$$\sum_{\substack{n=1 \\ \left(\frac{a}{p}\right)=1}}^{\infty} \left(\frac{n}{p}\right)_4 n^{-1} = \operatorname{Re}(L(1, \chi_{\omega})), \quad \sum_{\substack{n=1 \\ \left(\frac{a}{p}\right)=-1}}^{\infty} \left(\frac{\chi_{\omega}(n)}{i}\right) n^{-1} = \operatorname{Im}(L(1, \chi_{\omega})),$$

where

$$L(1, \chi_{\omega}) = \sum_{n=1}^{\infty} \chi_{\omega}(n) n^{-1}.$$

Example 1. $p = 5$. Here $a = 1$, $b = -2$, $\omega = 1 - 2i$, and part (a) of the Theorem gives

$$1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \cdots \\ = \sum_{\substack{n=1 \\ \left(\frac{a}{p}\right)=1}}^{\infty} \left(\frac{n}{5}\right)_4 n^{-1} \\ = -\frac{\pi}{50} \left(\sqrt{10 + 2\sqrt{5}} (F_{1-2i}(1) - F_{1-2i}(-1)) \right. \\ \left. - \sqrt{10 - 2\sqrt{5}} (F_{1-2i}(i) - F_{1-2i}(-i)) \right) \\ = \frac{\pi}{50} \left(3\sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}} \right) \\ \simeq 0.8648.$$

Part (b) of the Theorem gives

$$-\frac{1}{2} + \frac{1}{3} - \frac{1}{7} + \frac{1}{8} - \frac{1}{12} + \frac{1}{13} - \cdots = \sum_{\substack{n=1 \\ \left(\frac{a}{p}\right)=-1}}^{\infty} \left(\frac{\chi_{1-2i}(n)}{i}\right) n^{-1}$$

(continues)

$$\begin{aligned}
&= -\frac{\pi}{50} \left(-\sqrt{10-2\sqrt{5}}(F_{1-2i}(1) - F_{1-2i}(-1)) \right. \\
&\quad \left. -\sqrt{10+2\sqrt{5}}(F_{1-2i}(i) - F_{1-2i}(-i)) \right) \\
&= \frac{\pi}{50} \left(-3\sqrt{10-2\sqrt{5}} + \sqrt{10+2\sqrt{5}} \right) \\
&\simeq -0.2041.
\end{aligned}$$

Example 2. $p = 17$. Here $a = 1$, $b = 4$, $\omega = 1 + 4i$, and part (a) of the Theorem gives

$$\begin{aligned}
&1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{9} + \frac{1}{13} - \frac{1}{15} + \frac{1}{16} + \frac{1}{18} - \frac{1}{19} + \frac{1}{21} - \frac{1}{25} - \cdots \\
&= \sum_{\substack{n=1 \\ (\frac{n}{17})=1}}^{\infty} \left(\frac{n}{17} \right)_4 n^{-1} \\
&= \frac{1}{34} \left(-\sqrt{34-2\sqrt{17}}(LS_{1+4i}(1) - LS_{1+4i}(-1)) \right. \\
&\quad \left. +\sqrt{34+2\sqrt{17}}(LS_{1+4i}(i) - LS_{1+4i}(-i)) \right) \\
&= \frac{1}{17} \left(\sqrt{34-2\sqrt{17}} \log \left(4 \cos \frac{\pi}{17} \cos \frac{4\pi}{17} \right) \right. \\
&\quad \left. +\sqrt{34+2\sqrt{17}} \log \left(4 \cos \frac{3\pi}{17} \cos \frac{5\pi}{17} \right) \right) \\
&\simeq 0.5927.
\end{aligned}$$

Part (b) of the Theorem gives

$$\begin{aligned}
&-\frac{1}{3} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} - \frac{1}{12} - \frac{1}{14} - \frac{1}{20} - \frac{1}{22} + \frac{1}{23} + \frac{1}{30} + \cdots \\
&= \sum_{\substack{n=1 \\ (\frac{n}{17})=-1}}^{\infty} \left(\frac{\chi_{1+4i}(n)}{i} \right) n^{-1} \\
&= \frac{1}{34} \left(\sqrt{34+2\sqrt{17}}(LS_{1+4i}(1) - LS_{1+4i}(-1)) \right. \\
&\quad \left. +\sqrt{34-2\sqrt{17}}(LS_{1+4i}(i) - LS_{1+4i}(-i)) \right) \\
&= \frac{1}{17} \left(-\sqrt{34+2\sqrt{17}} \log \left(4 \cos \frac{\pi}{17} \cos \frac{4\pi}{17} \right) \right. \\
&\quad \left. +\sqrt{34-2\sqrt{17}} \log \left(4 \cos \frac{3\pi}{17} \cos \frac{5\pi}{17} \right) \right) \\
&\simeq -0.1936.
\end{aligned}$$

Proof. The quartic Gauss sum $G(\chi_\omega)$ is defined by

$$(10) \quad G(\chi_\omega) = \sum_{r=1}^{p-1} \chi_\omega(r) \exp(2\pi i r/p).$$

The value of $G(\chi_\omega)$ was conjectured by Loxton [2] in 1978 and proved by Mathews [3] in 1979. Mathews showed that

$$(11) \quad G(\chi_\omega) = i^{(p+2b+3)/4} \left(\frac{2|b|}{|a|} \right) p^{1/4} \omega^{1/2},$$

where the large brackets denote the usual Jacobi symbol and it is understood throughout that fractional powers take their principal values. We note that

$$G(\bar{\chi}_\omega) = (-1)^{(p-1)/4} \overline{G(\chi_\omega)},$$

so that by (11) we have

$$(12) \quad G(\bar{\chi}_\omega) = (-1)^{(p+3)/4} i^{(p+2b+3)/4} \left(\frac{2|b|}{|a|} \right) p^{1/4} \bar{\omega}^{1/2}.$$

For convenience we set

$$(13) \quad X = (-1)^{(p+2b+7)/8}, \quad Y = \frac{b}{|b|}, \quad Z = \left(\frac{2|b|}{|a|} \right),$$

$$(14) \quad \alpha = \sqrt{2p + 2a\sqrt{p}}, \quad \beta = \sqrt{2p - 2a\sqrt{p}},$$

so that

$$(15) \quad \alpha\beta = 2|b|p^{1/2},$$

$$(16) \quad \omega^{1/2} - \bar{\omega}^{1/2} = iY\beta/p^{1/4},$$

$$(17) \quad \omega^{1/2} + \bar{\omega}^{1/2} = \alpha/p^{1/4}.$$

For $m \not\equiv 0 \pmod{p}$, appealing to (11), (12), (13), (14), (16), (17), we obtain

$$(18) \quad \begin{aligned} & G(\chi_\omega)\bar{\chi}_\omega(m) + G(\bar{\chi}_\omega)\chi_\omega(m) \\ &= i^{(p+2b+3)/4} Z p^{1/4} (\bar{\chi}_\omega(m)\omega^{1/2} + (-1)^{(p+3)/4} \chi_\omega(m)\bar{\omega}^{1/2}) \\ &= \begin{cases} \chi_\omega(m)XYZ\beta, & \text{if } \chi_\omega(m) = \pm 1, p \equiv 1 \pmod{8}, \\ -i\chi_\omega(m)XZ\alpha, & \text{if } \chi_\omega(m) = \pm 1, p \equiv 5 \pmod{8}, \\ i\chi_\omega(m)XZ\alpha, & \text{if } \chi_\omega(m) = \pm i, p \equiv 1 \pmod{8}, \\ -\chi_\omega(m)XYZ\beta, & \text{if } \chi_\omega(m) = \pm i, p \equiv 5 \pmod{8}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & G(\chi_\omega)\bar{\chi}_\omega(m) - G(\bar{\chi}_\omega)\chi_\omega(m) \\ &= i^{(p+2b+3)/4} Z p^{1/4} (\bar{\chi}_\omega(m)\omega^{1/2} - (-1)^{(p+3)/4} \chi_\omega(m)\bar{\omega}^{1/2}) \end{aligned}$$

$$(19) \quad = \begin{cases} -i\chi_{\omega}(m)XZ\alpha, & \text{if } \chi_{\omega}(m) = \pm 1, \ p \equiv 1 \pmod{8}, \\ \chi_{\omega}(m)XYZ\beta, & \text{if } \chi_{\omega}(m) = \pm 1, \ p \equiv 5 \pmod{8}, \\ -\chi_{\omega}(m)XYZ\beta, & \text{if } \chi_{\omega}(m) = \pm i, \ p \equiv 1 \pmod{8}, \\ i\chi_{\omega}(m)XZ\alpha, & \text{if } \chi_{\omega}(m) = \pm i, \ p \equiv 5 \pmod{8}. \end{cases}$$

Next, from the work of Dirichlet (see for example [1, Theorem 1.3, p. 137]), we have

$$(20) \quad L(1, \chi_{\omega}) = \begin{cases} -(G(\chi_{\omega})/p) \sum_{m=1}^{p-1} \bar{\chi}_{\omega}(m) \log \left(\sin \frac{m\pi}{p} \right), & \text{if } p \equiv 1 \pmod{8}, \\ (\pi i G(\chi_{\omega})/p^2) \sum_{m=1}^{p-1} m \bar{\chi}_{\omega}(m), & \text{if } p \equiv 5 \pmod{8}, \end{cases}$$

and

$$(21) \quad L(1, \bar{\chi}_{\omega}) = \begin{cases} -(G(\bar{\chi}_{\omega})/p) \sum_{m=1}^{p-1} \chi_{\omega}(m) \log \left(\sin \frac{m\pi}{p} \right), & \text{if } p \equiv 1 \pmod{8}, \\ (\pi i G(\bar{\chi}_{\omega})/p^2) \sum_{m=1}^{p-1} m \chi_{\omega}(m), & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

Hence, from (20) and (21), we obtain

$$\begin{aligned} \sum_{\substack{n=1 \\ (\frac{n}{p})=1}}^{\infty} \left(\frac{n}{p} \right)_4 n^{-1} &= \frac{1}{2} \sum_{n=1}^{\infty} (\chi_{\omega}(n) + \bar{\chi}_{\omega}(n)) n^{-1} \\ &= \frac{1}{2} (L(1, \chi_{\omega}) + L(1, \bar{\chi}_{\omega})) \\ &= \begin{cases} \frac{-1}{2p} \sum_{m=1}^{p-1} (G(\chi_{\omega}) \bar{\chi}_{\omega}(m) + G(\bar{\chi}_{\omega}) \chi_{\omega}(m)) \log \left(\sin \frac{m\pi}{p} \right), & \text{if } p \equiv 1 \pmod{8}, \\ \frac{\pi i}{2p^2} \sum_{m=1}^{p-1} (G(\chi_{\omega}) \bar{\chi}_{\omega}(m) + G(\bar{\chi}_{\omega}) \chi_{\omega}(m)) m, & \text{if } p \equiv 5 \pmod{8}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{n=1 \\ (\frac{n}{p})=-1}}^{\infty} \left(\frac{\chi_{\omega}(n)}{i} \right) n^{-1} &= \frac{1}{2i} \sum_{n=1}^{\infty} (\chi_{\omega}(n) - \bar{\chi}_{\omega}(n)) n^{-1} \\ &= \frac{1}{2i} (L(1, \chi_{\omega}) - L(1, \bar{\chi}_{\omega})) \\ &= \begin{cases} \frac{i}{2p} \sum_{m=1}^{p-1} (G(\chi_{\omega}) \bar{\chi}_{\omega}(m) - G(\bar{\chi}_{\omega}) \chi_{\omega}(m)) \log \left(\sin \frac{m\pi}{p} \right), & \text{if } p \equiv 1 \pmod{8}, \\ \frac{\pi}{2p^2} \sum_{m=1}^{p-1} (G(\chi_{\omega}) \bar{\chi}_{\omega}(m) - G(\bar{\chi}_{\omega}) \chi_{\omega}(m)) m, & \text{if } p \equiv 5 \pmod{8}. \end{cases} \end{aligned}$$

The theorem now follows using (18) and (19).

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