HYPERCOMPLETIONS OF RIESZ SPACES

WOLFGANG FILTER

(Communicated by John B. Conway)

ABSTRACT. It is shown that each Riesz space with separating order continuous dual can be embedded in a unique “ e -hypercompletion,” where e is a fixed weak unit of the extended order continuous dual.

1. INTRODUCTION

In [8] I investigated—in terms of an internal characterization—those Riesz spaces which can be represented as bands of spaces of measures; I call these spaces hypercomplete. The present paper is devoted to the question of how far Archimedean Riesz spaces can be embedded in a “hypercompletion.” It follows already from [7, Theorem A] that each Riesz space L with separating order continuous dual $L_n^\sim$ (i.e. $(L_n^\sim)^0 = \{0\}$) embeds order densely in some hypercomplete Riesz space M , but uniqueness is not yet guaranteed. The main result given here is that each Riesz space L with $(L_n^\sim)^0 = \{0\}$ can be embedded order densely in a unique “ e -hypercompletion” where e is some weak unit of the extended order continuous dual $\Gamma(L)$ of L . But we will see in an example that, in general, dependence on the weak unit e cannot be avoided.

2. RESULTS

L always denotes an Archimedean Riesz space.

For convenience, the definitions of extended order continuous dual and hypercompleteness follow.

Let Φ be the set of all order dense ideals of L . Then $\Gamma(L) := \bigcup_{I \in \Phi} I_n^\sim$ is the *extended order continuous dual* of L (see [10]), where $\xi \in I_n^\sim$ and $\eta \in J_n^\sim$ are identified if they coincide on $I \cap J$. With algebraic and order structure defined by using representatives, $\Gamma(L)$ is a universally complete Riesz space [10, 1.5], and for each $\xi \in \Gamma(L)$ there exists a greatest $L[\xi] \in \Phi$ such that $\xi \in L[\xi]_n^\sim$ [10, 1.3].

If e is a weak unit of $\Gamma(L)$ and $R \subset L_n^\sim$ is a set of components of e such that for each upward-directed family (x_l) from L^+ for which $\sup g(x_l) < \infty$

Received by the editors September 1, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46A40.

©1990 American Mathematical Society
 0002-9939/90 \$1.00 + \$.25 per page

for every $g \in R$ there exists $\sup x_l$ in L , then call (e, R) an *hc-pair* of L . Call L *hypercomplete* if there exists an *hc-pair* of L .

In the sequel let e be a weak unit of $\Gamma(L)$. I denote by R_e the set of all components of e in $L_n^\sim$.

Before reading the appropriate definition, note that it would not be senseful to call a Riesz space M *e-hypercompletion* of L if L is order dense in M and e is the first component of an *hc-pair* of M , since in this case M would not be unique: If $\mathcal{M}(\mathcal{R})$ denotes the set of all real-valued σ -additive measures on some δ -ring $\mathcal{R}$ of subsets of a set X , and $\mathcal{M}_b(\mathcal{R})$ denotes the bounded measures on $\mathcal{R}$, then for $L = \mathcal{M}_b(\mathcal{R})$ and $e = \mathbf{1}_X$, both $M_1 = \mathcal{M}_b(\mathcal{R})$ and $M_2 = \mathcal{M}(\mathcal{R})$ would satisfy this bad definition. One has to impose an extra condition for this reason.

If M is an Archimedean Riesz space and if $\phi: L \rightarrow M$ is an injective Riesz homomorphism such that ϕL is order dense in M , I denote by $\psi_{M,\phi}$ the canonical Riesz isomorphism

$$\Gamma(M) \rightarrow \Gamma(L), \xi \mapsto \xi \circ \phi|_{\phi^{-1}(M[\xi])}$$

([10, 2.6]).

Definition 1. A hypercomplete Riesz space M is called *e-hypercompletion* of L if there exists an injective Riesz homomorphism $\phi: L \rightarrow M$ such that

1. ϕL is an order dense Riesz subspace of M ;
2. we have $\psi_{M,\phi}^{-1}(g) \in M_n^\sim$ for all $g \in R_e$, and $(\psi_{M,\phi}^{-1}(e), \psi_{M,\phi}^{-1}(R_e))$ is an *hc-pair* of M .

Roughly, M is an *e-hypercompletion* of L iff L is order densely embedded in M , $R_e \subset M_n^\sim$, and (e, R_e) is an *hc-pair* of M .

It follows from condition 1 and [8, 2.3] that only spaces with $(L_n^\sim)^0 = \{0\}$ can possess an *e-hypercompletion*. For this reason, we assume $L_n^\sim$ to be separating from now on.

If Y is a locally compact Stonian space, let $\mathcal{M}(Y)$ denote the set of normal Radon measures on Y .

Proposition 2. If M_1, M_2 are *e-hypercompletions* of L , then M_1 and M_2 are canonically Riesz isomorphic, i.e. if for $k \in \{1, 2\}$, $\phi_k: L \rightarrow M_k$ satisfies the definition above, then there exists a Riesz isomorphism $v: M_1 \rightarrow M_2$ such that $v \circ \phi_1 = \phi_2$.

Proof. Set $\psi_k := \psi_{M_k, \phi_k}$. By the Ogasawara-Maeda representation theorem there exist a compact hyperstonian space Z and a Riesz isomorphism $u: \Gamma(L) \rightarrow C_\infty(Z)$ with $u(e) = 1_Z$. Then $u \circ \psi_k: \Gamma(M_k) \rightarrow C_\infty(Z)$ is a Riesz isomorphism with $u \circ \psi_k(\psi_k^{-1}(e)) = 1_Z$. Set

$$Y_k := \bigcup_{g \in R_e} \text{supp } u \circ \psi_k(\psi_k^{-1}(g))$$

$$\hat{u}_k: \Gamma(M_k) \rightarrow C_\infty(Y_k), \xi \mapsto (u \circ \psi_k)(\xi)|_{Y_k}.$$

By [7, Theorem A], there exists a Riesz isomorphism $v_k: M_k \rightarrow \mathcal{M}(Y_k)$ such that $(Y_k, \hat{u}_k, v_k)$ is a $\psi_k^{-1}(e)$ -mr of M_k to which $\psi_k^{-1}(R_e)$ is associated; observe that v_k is onto by the hypercompleteness of M_k . But $Y_1 = Y_2$, and hence $v := v_2^{-1} \circ v_1$ is a Riesz isomorphism.

Fix $x \in L$. Then for all $\xi \in \Gamma(L)$ for which $\xi(x)$ is defined:

$$\begin{aligned} \int u\xi|_{Y_1} dv_1(\phi_1 x) &= \int \hat{u}_1(\psi_1^{-1}\xi) dv_1(\phi_1 x) = (\psi_1^{-1}\xi)(\phi_1 x) = \xi(x) \\ &= \cdots = \int u\xi|_{Y_2} dv_2(\phi_2 x). \end{aligned}$$

Hence $v_1(\phi_1 x) = v_2(\phi_2 x)$ and thus $v \circ \phi_1 = \phi_2$. $\square$

Let G_e denote the ideal of $L_n^\sim$ generated by R_e .

Theorem 3. $(G_e)_n^\sim$ is the e -hypercompletion of L .

Proof. By [7, Theorem A] there exists an e -mr $(Y, \hat{u}, v)$ of L to which R_e is associated. Then $(1_Y, \{\hat{u}g | g \in R_e\})$ is an hc -pair of $\mathcal{M}(Y)$, which implies that $\mathcal{M}(Y)$ is an e -hypercompletion of L .

Since obviously $G_e = C_c(Y)$ (the set of continuous functions on Y with compact support), it follows from the Riesz representation theorem that $(G_e)_n^\sim = \mathcal{M}(Y)$.

The uniqueness was proved in Proposition 2. $\square$

Of course, L is its own e -hypercompletion if (e, R_e) is an hc -pair of L .

As we have seen in the proof of the preceding theorem, the e -hypercompletion of L can always be realized as the space $\mathcal{M}(Y)$, for some locally compact hyperstonian space Y . Another beautiful description of it in terms of extended $L^1(\mu)$ -spaces—where μ is a normal measure on a compact Stonian space—was obtained by Y. A. Abramovich [15].

Copying the proof of Theorem 3, we get

Corollary 4. If $R \subset R_e$ with $e = \sup_{g \in R} g$, then $(G_R)_n^\sim$ is hypercomplete, where G_R stands for the ideal of $L_n^\sim$ generated by R . $\square$

But $(G_R)_n^\sim = (G_e)_n^\sim$ need not always hold: If $L = \mathcal{M}_b(\mathcal{R})$, $e = \dot{1}_X$, $R = \{\dot{1}_A | A \in \mathcal{R}\}$, then $(G_e)_n^\sim = \mathcal{M}_b(\mathcal{R})$ while $(G_R)_n^\sim = \mathcal{M}(\mathcal{R})$.

The following example shows that, in general, to different e 's there belong different e -hypercompletions.

Example. Let $L := \ell^\infty$. Then $L_n^\sim = \ell^1$ and $\Gamma(L) = \mathbb{R}^{\mathbb{N}}$. For each weak unit e of $\mathbb{R}^{\mathbb{N}}$, set

$$\begin{aligned} \mathcal{A}(e) &:= \left\{ A \subset \mathbb{N} \mid \sum_{n \in A} e(n) < \infty \right\}, \\ M_e &:= \left\{ h \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n \in A} |h(n)|e(n) < \infty \text{ for all } A \in \mathcal{A}(e) \right\}. \end{aligned}$$

Then M_e is the e -hypercompletion of L , where the embedding map ϕ is just the identity map on L .

It follows that if $v: M_{e_1} \rightarrow M_{e_2}$ is a Riesz isomorphism preserving the embedding of L , we must have $v|_L = id$, and hence, by the order continuity of v , $v = id$, which implies $M_{e_1} = M_{e_2}$. But if, for example, $e_1, e_2 \in L_n^\sim$, then the condition $M_{e_1} = M_{e_2}$ is easily seen to be equivalent to the existence of positive reals C, D with $Ce_1 \leq e_2 \leq De_1$, which allows us to construct many nonisomorphic e -hypercompletions of L . $\square$

Let us briefly consider the situation where $L_n^\sim$ possesses weak units.

If $e \in L_n^\sim$, then e is the greatest element of R_e , and since $R_e \subset ((G_e)_n^\sim)_n^\sim$ by Theorem 3, we have

Proposition 5. *If $e \in L_n^\sim$ then $(e, \{e\})$ is an hc-pair of $(G_e)_n^\sim$. $\square$*

Let M be a Dedekind complete Riesz space, and denote, for $x \in M$, by $S(x)$ the ideal of M generated by x . Then, if f, g are weak units of M , the Ogasawara-Maeda theorem implies the existence of a Riesz isomorphism $\phi: S(f) \rightarrow S(g)$ with $\phi f = g$.

An immediate consequence of this remark is that for weak units e_1, e_2 of $L_n^\sim$, the corresponding e -hypercompletions are Riesz isomorphic (but of course not canonically, i.e. not preserving the embedding of L):

Proposition 6. *If e_1, e_2 are weak units of $L_n^\sim$, then G_{e_1} and G_{e_2} are Riesz isomorphic, and hence the same holds for $(G_{e_1})_n^\sim$ and $(G_{e_2})_n^\sim$. $\square$*

We can even say that an e -hypercompletion belonging to $e \in L_n^\sim$ is in some sense minimal. Consider another weak unit f of $\Gamma(L)$. Then $e_1 := \inf\{e, f\}$ is a weak unit of $L_n^\sim$, and since $S(f)$ is isomorphic to G_{e_1} (the remark above applied to $\Gamma(L)$), G_f embeds order densely in G_{e_1} , which implies that $(G_{e_1})_n^\sim$ embeds in $(G_f)_n^\sim$, and hence $(G_e)_n^\sim$ embeds in $(G_f)_n^\sim$, too.

3. AN APPLICATION

I now briefly sketch an application; a more detailed presentation will be given in a subsequent paper.

Several authors (cf. [1-5, 9, 11-14]) have studied so-called Hellinger integrals to get integral representations of the duals of spaces of finitely additive or σ -additive measures; here a real-valued function ϕ , defined on some ring of sets, $\mathcal{R}$, is called Hellinger integrable with respect to a measure μ having domain $\mathcal{R}$ if $\lim \sum_{A \in \mathcal{A}} \phi(A) \mu(A)$ exists in $\mathbb{R}$, where $\mathcal{A}$ runs through the set of partitions of $\mathcal{R}$, directed under refinement. But the representation theorems these authors get are not very satisfying in some sense, because set theoretical assumptions have to be made (see e.g. [12]), and one must necessarily restrict to "small" spaces of measures as Keisler has shown [9, Theorem 4]. A method proposed here to overcome these difficulties is to work with generalized parti-

tions as described below; this method works not only in spaces of measures but in Riesz spaces L with separating dual $L_n^\sim$:

$\mathcal{P}$ denotes the set of all countable disjoint systems from R_e . Setting $g_{\mathcal{A}} := \sup_{g \in \mathcal{A}} g$ for $\mathcal{A} \in \mathcal{P}$, one can define an order relation on $\mathcal{P}$ by

$$\mathcal{A} \leq \mathcal{B} : \Leftrightarrow g_{\mathcal{A}} \leq g_{\mathcal{B}}, \text{ and for all } h \in \mathcal{B} \text{ either} \\ \inf\{h, g_{\mathcal{A}}\} = 0 \text{ or there is } g \in \mathcal{A} \text{ with } h \leq g.$$

Let $\mathcal{F}$ be the section filter of $\mathcal{P}$ with respect to $\leq$. For $x \in L$ and $\phi: R_e \rightarrow \mathbb{R}$ set $\phi_x(\mathcal{A}) := \sum_{g \in \mathcal{A}} \phi(g)g(x)$ if the series is summable, and $\phi_x(\mathcal{A}) := \infty$, otherwise ($\mathcal{A} \in \mathcal{P}$). One can call ϕ x -integrable if $\phi_x(\mathcal{F})$ converges in $\mathbb{R}$; in this case, set

$$\int \phi dx := \lim \phi_x(\mathcal{F}).$$

The following representation theorem may serve to justify the definition of generalized Hellinger integral just given (observe that this theorem applies in particular to spaces of measures since they are hypercomplete by [8, 2.5]):

Theorem 7. *For each $\xi \in \Gamma(L)$ which is extendable to $M[e]$ (where M denotes the e -hypercompletion of L), there exists a bounded map $\phi: R_e \rightarrow \mathbb{R}$ such that ϕ is x -integrable and $\xi(x) = \int \phi dx$ for all $x \in L[\xi]$.*

Proof (idea of). Let $(Y, \hat{u}, v)$ be the e -mr of M to which $R_e \subset M_n^\sim$ is associated; then $vM = \mathcal{M}(Y)$, by [7, Theorem A] and the hypercompleteness of M .

For $g \in R_e$, $g \neq 0$, set $\phi(g) := \hat{u}\xi(y_g)$, where y_g is a point of $\text{supp } \hat{u}g$ in which $|\hat{u}\xi|$ attains its minimum on $\text{supp } \hat{u}g$; for $g = 0$, set $\phi(g) := 0$.

Since $\hat{u}\xi$ is defined on $\mathcal{M}_b(Y) = v(M[e])$, $\hat{u}\xi$ is bounded by [6, 1.6.1b)], and hence ϕ is likewise. The rather technical verification that ϕ has the required properties is omitted here. $\square$

Every $\xi \in \Gamma(L)$ can be represented in the same way, but ϕ will no longer be bounded in general. The advantage of the boundedness of ϕ is that in this case one obtains the same Hellinger integral when working only with finite disjoint systems from R_e (at least for $x \in L[e]$).

ACKNOWLEDGMENT

I am very grateful to the referee for a number of useful comments.

REFERENCES

1. W. D. L. Appling, *Concerning a class of linear transformations*, J. London Math. Soc. **44** (1969), 385–396.
2. ———, *An isomorphism and isometry theorem for a class of linear functionals*, Trans. Amer. Math. Soc. **199** (1974), 131–140.
3. W. C. Bell, *A decomposition of additive set functions*, Pacific J. Math. **72** (1977), 305–311.
4. ———, *Approximate Hahn decompositions, uniform absolute continuity and uniform integrability*, J. Math. Anal. Appl. **80** (1981), 393–405.

5. W. C. Bell and M. Keisler, *A characterization of the representable Lebesgue decomposition properties*, Pacific J. Math. **83** (1979), 185–186.
6. C. Constantinescu, *Duality in measure theory*, Lecture Notes in Math., vol. 796, Springer-Verlag, Berlin, Heidelberg, and New York, 1980.
7. W. Filter, *Atomical and atomfree elements of a Riesz space*, Arch. Math. **52** (1989), 580–587.
8. ———, *Hypercomplete Riesz spaces*, Atti Sem. Mat. Fis. Univ. Modena (to appear).
9. M. Keisler, *Integral representation for elements of the dual of $ba(S, \Sigma)$* , Pacific J. Math. **83** (1979), 177–183.
10. W. A. J. Luxemburg and J. J. Masterson, *An extension of the concept of the order dual of a Riesz space*, Canad. J. Math. **19** (1967), 488–498.
11. J. S. MacNerney, *Finitely additive set functions*, Houston J. Math. **6** (1980).
12. R. D. Mauldin, *A representation theorem for the second dual of $C[0, 1]$* , Studia Math. **46** (1973), 197–200.
13. ———, *The continuum hypothesis, integration and duals of measure spaces*, Illinois J. Math. **19** (1975), 33–40.
14. ———, *Some effects of set-theoretical assumptions in measure theory*, Adv. in Math. **27** (1978), 45–62.
15. Y. A. Abramovich, *A special class of vector lattices and its application to hypercomplete spaces*, preprint.

MATHEMATIK, ETH-ZENTRUM, CH-8092 ZÜRICH, SWITZERLAND