FUNCTIONAL CHARACTERIZATIONS OF BAIRE SPACES

ĽUBICA HOLÁ

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ABSTRACT. We give functional characterizations of Baire spaces in the class of topological spaces with a σ -locally finite almost-base. This answers positively a question asked in [5] for topological spaces with a σ -locally finite almost-base.

INTRODUCTION

In [5] is posed the following question:

Let E be a space such that a certain graph theorem holds for E and all suitable F and mappings $f: E \rightarrow F$. Must E then be a Baire space?

It is shown in [5] that the answer is positive if E is assumed to be metrizable. We give a simple proof that the answer is also positive if E is a topological space with a σ -locally finite almost-base.

Let E, F be topological spaces. A mapping f from E to F is called nearly continuous if for every $x \in E$ and every neighbourhood U of $f(x)$ the set $f^{-1}(U)$ is a neighbourhood of x . (Compare [1], [6].)

Following Bourbaki, we will call a topological space F Polish, if F is separable and metrizable by a complete metric.

A family $\mathcal{V}$ of open subsets of E is called an almost-base of E if for every nonempty open subset U of E there is a nonempty set $V \in \mathcal{V}$ such that $V \subset U$. (See [3].)

Recall that $f: E \rightarrow F$ is called Borel measurable of class one if the preimages of open sets are F_σ -sets.

Lemma. *Let E be a topological space and F be a metrizable separable space. If $f: E \rightarrow F$ is a Borel measurable function of class one, then the graph of f is a G_δ -set in $E \times F$.*

Proof. Let $\mathbf{R}$ be the set of the real numbers. Define $\varphi: E \times F \rightarrow \mathbf{R}$ by $\varphi(x, y) = d(f(x), y)$, where d is a metric in F . We prove that φ is a Borel measurable function of class one. It is sufficient to prove that for every $a \in \mathbf{R}$, $a > 0$ the sets $\{(x, y): \varphi(x, y) > a\}$ and $\{(x, y): \varphi(x, y) < a\}$ are F_σ -sets.

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Let $(y_n: n = 1, 2, \dots)$ be a countable dense set in F . For each $z \in F$ and each $r > 0$ put $B(z, r) = \{y \in F: d(z, y) < r\}$. Fix $a > 0$. Set $T_k = \{(y_i, y_j): d(y_i, y_j) > (a + 1/2k)\}$, $k = 1, 2, \dots$. It is easy to verify that

$$\{(x, y): \varphi(x, y) > a\} = \bigcup_k \bigcup_{(y_i, y_j) \in T_k} f^{-1}(B(y_i, 1/4k)) \times B(y_j, 1/4k)$$

and

$$\{(x, y): \varphi(x, y) < a\} = \bigcup_k \bigcup_{y_n} f^{-1}(B(y_n, a - 1/k)) \times B(y_n, 1/k).$$

Since f is Borel measurable function of class one, the above equalities guarantee that the sets $\{(x, y): \varphi(x, y) > a\}$ and $\{(x, y): \varphi(x, y) < a\}$ are F_σ -sets. Thus for each $a, b \in \mathbb{R}$, $a, b > 0$ $\{(x, y): a < \varphi(x, y) < b\}$ is an F_σ -set. Since $\mathbb{R}$ is a second countable, φ is a Borel measurable function of class one. Thus $\{(x, y): y = f(x)\} = \{(x, y): \varphi(x, y) = 0\} = \varphi^{-1}(\{0\})$ is a G_δ -set.

The main theorems follow.

Theorem 1. *Let E be a topological space with σ -locally finite almost-base and such that every singleton is a G_δ -set. Let F be a Polish space with at least two elements. The following statements are equivalent:*

- (1) E is a Baire space;
- (2) every nearly continuous mapping $f: E \rightarrow F$ which is Borel measurable of class one is continuous;
- (3) every nearly continuous mapping $f: E \rightarrow F$ whose graph is a G_δ -set is continuous.

Proof. (1) implies (2) and (3) by [5, Theorems 3 and 1] respectively. By the Lemma, (3) implies (2). It remains only to prove that (2) implies (1).

Assume that E is not Baire and choose a nonempty open set U which is of the first category. Let $C_1, C_2, \dots$ be a sequence of nowhere dense sets with $\bigcup\{C_n: n \in \mathbb{N}\} = U$. Let $\mathcal{V}_n$, $n \in \mathbb{N}$ be locally finite sets of nonempty open sets such that $\mathcal{V} = \bigcup\{\mathcal{V}_n: n \in \mathbb{N}\}$ is almost-base of E . For every $n \in \mathbb{N}$ put $\mathcal{B}_n = \{U \cap V: V \in \mathcal{V}_n\}$. We construct sequences $(G(n): n \in \mathbb{N})$, $(H(n): n \in \mathbb{N})$ of subsets of E with the following properties:

- (a) $G(n), H(n)$ are nowhere dense sets;
- (b) $G(n)$ are closed G_δ -sets, $H(n)$ are F_σ -sets;
- (c) for all $B \in \mathcal{B}_n$ $H(n) \cap B \neq \emptyset$ and $G(n) \cap B \neq \emptyset$;
- (d) $G(n) \cap (\bigcup\{H(i): i \leq n\}) = \emptyset$ and $H(n) \cap (\bigcup\{G(i): i \leq n\}) = \emptyset$, for $n \in \mathbb{N}$;
- (e) $U \subset \bigcup\{G(n): n \in \mathbb{N}\} \cup \bigcup\{H(n): n \in \mathbb{N}\}$.

Let $n = 1$. For every $B \in \mathcal{B}_1$ choose $x(B) \in B$ and define $G(1) = \{x(B): B \in \mathcal{B}_1\}$. Since $\mathcal{B}_1$ is locally finite and every singleton is a G_δ -set, $G(1)$ is a closed nowhere dense G_δ -set. For every $B \in \mathcal{B}_1$ choose $y(B) \in B \setminus G(1)$.

It is possible since every set from $\mathcal{B}_1$ is open. Define $H(1) = \{y(B) : B \in \mathcal{B}_1\} \cup (\overline{C}_1 \setminus G(1))$. It is easy to verify that $H(1)$ is a nowhere dense F_σ -set.

Suppose now that $G(i)$, $H(i)$ have been defined for $i < n$ such that (a)–(d) are satisfied. Put $\mathcal{B}'_n = \{B \setminus \bigcup\{G(i) \cup H(i) : i < n\} : B \in \mathcal{B}_n\}$. For every $B \in \mathcal{B}'_n$ choose $x(B) \in B$ and define $G(n) = \{x(B) : B \in \mathcal{B}'_n\}$. Then $G(n)$ is a closed nowhere dense G_δ -set. For every $B \in \mathcal{B}'_n$ choose $y(B) \in B \setminus G(n)$ and define $H(n) = \{y(B) : B \in \mathcal{B}'_n\} \cup (\overline{C}_n \setminus \bigcup\{G(i) : i \leq n\})$. Then (a)–(e) are satisfied.

Suppose now that $G(n)$, $H(n)$, $n \in \mathbb{N}$ have been constructed and define $f: E \rightarrow F$ by $f(\bigcup\{G(n) : n \in \mathbb{N}\}) = a$ and $f(E \setminus \bigcup\{G(n) : n \in \mathbb{N}\}) = b$, where a, b are two different points from F . Since $f^{-1}(b) = (E \setminus U) \cup \bigcup\{H(n) : n \in \mathbb{N}\}$ is an F_σ -set, f is Borel measurable of class one. We prove that it is nearly continuous. Let $x \in \bigcup\{G(n) : n \in \mathbb{N}\}$. Since $\bigcup\{G(n) : n \in \mathbb{N}\}$ is a dense subset of U , f is nearly continuous at x . Let $x \in E \setminus \bigcup\{G(n) : n \in \mathbb{N}\}$. The set $E \setminus \bigcup\{G(n) : n \in \mathbb{N}\}$ is dense in E , i.e., f is nearly continuous at x . But f is not continuous.

Theorem 2. *Let E be a T_2 topological space with a σ -discrete base. Let F be an infinite, second countable T_2 topological space. The following statements are equivalent:*

- (1) E is a Baire space;
- (2) for every mapping $f: E \rightarrow F$ there is a dense set D in E such that the restriction $f|D$ is continuous.

Proof. Let E be a Baire space. Then by [6] there is a dense set A in E such that $f|A$ is nearly continuous. Put $g = f|A$. By [4] there is a dense set D in A such that $g|D$ is continuous. Then D is dense in E and $f|D$ is continuous.

Now assume that E is not Baire and choose a nonempty open set U which is of the first category. Let $C_1, C_2, \dots$ be a sequence of mutually disjoint nowhere dense sets with $\bigcup\{C_n : n \in \mathbb{N}\} = U$. Let L be an infinite discrete set of different points from F and let $(c_n : n \geq 0)$ be an enumeration of L . Define $f: E \rightarrow F$ by $f(C_n) = c_n$, $n \geq 1$, and $f(E \setminus U) = c_0$. There is no set D dense in E for which the restriction $f|D$ is continuous. Suppose that there is a dense set D in E such that the restriction $f|D$ is continuous. Choose $x \in D \cap U$. There is $n \geq 1$ such that $f(x) = c_n$. Since L is a discrete set and $f|D$ is continuous at x there is an open neighbourhood V of x in E such that $f(V \cap U \cap D) = c_n$. Thus $C_n = f^{-1}(c_n) \supset V \cap U \cap D$, i.e., $\overline{C}_n \supset V \cap U$ and that is a contradiction since C_n is nowhere dense.

Theorem 2 is a generalization of a known result from [1].

REFERENCES

1. J. C. Bradford and C. Goffman, *Metric spaces in which Blumberg's Theorem holds*, Proc. Amer. Math. Soc. **11** (1960), 667–670.
2. T. Byczkowski and R. Pol, *On the closed graph and open mapping theorems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **24** (1976), 723–726.

3. Z. Frolik, *Baire spaces and some generalizations of complete metric spaces*, Czech. Math. J. **11** (1961), 237–248.
4. Ľ. Holá, *A remark on almost continuous multifunctions*, submitted in Mathematica Slovaca.
5. D. Noll, *Baire spaces and graph theorems*, Proc. Amer. Math. Soc. **96** (1986), 141–151.
6. M. Wilhelm, *Almost lower semicontinuous multifunctions and the Souslin-graph theorem*, Comment. Math. Univ. Carolin. **23** (1982), 147–158.

KADNÁROVÁ 4, 831 05 BRATISLAVA, ČSSR