HOMOGENEITY AND CANTOR MANIFOLDS

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ABSTRACT. Some consequences of generalized homogeneity are observed in dimension theory of metrizable spaces. In particular, if X is a connected, locally compact, metric space which is homogeneous with respect to open 0-dimensional mappings and if $\dim X = n \geq 1$ ($\dim X = \infty$), then no subset of dimension $\leq n - 2$ (respectively, of a finite dimension) separates X . Thus, homogeneous continua are Cantor manifolds.

A space X is said to be homogeneous with respect to a class M of mappings of X into X if for each pair of points x, y of X there exists a surjection $f \in M$ such that $f(x) = y$. If M is the class $H(X)$ of all autohomeomorphisms of X , then X is called homogeneous.

Many results concerning generalized homogeneity with respect to Borel classes of mappings have been obtained in recent years (e.g., see [2]). In this paper we observe some effects of generalized homogeneity in dimension theory.

Throughout the paper all spaces are assumed to be metrizable and all mappings are continuous. By $\dim X$ we understand the covering dimension of a space X .

A space X is called countable-dimensional (strongly countable-dimensional) if X can be represented as the union of a sequence $X_1, X_2, \dots$ of subspaces (closed subspaces) such that $\dim X_i < \infty$, $i = 1, 2, \dots$ [5].

A space X is said to be A -weakly infinite-dimensional (S -weakly infinite-dimensional) if for every sequence $(A_1, B_1), (A_2, B_2), \dots$ of pairs of disjoint closed subsets of X there exist closed sets $C_1, C_2, \dots$ such that C_n is a partition between A_n and B_n and $\bigcap_{n=1}^{\infty} C_n = \emptyset$ ($\bigcap_{n=1}^k C_n = \emptyset$ for some k). If X is not A - (S -) weakly infinite-dimensional, then X is called A - (S -) strongly infinite-dimensional (see [1, pp. 528–529] or [5, p. 33]). We use the abbreviations A -w.i.d., A -s.i.d., S -w.i.d., S -s.i.d. For compact spaces the notions of A -w.i.d. and S -w.i.d. (A -s.i.d. and S -s.i.d.) spaces coincide and we use the abbreviation w.i.d. (s.i.d.).

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It is known that connected, locally compact, metrizable spaces are separable (see [3, p. 241]).

The space X^X of all mappings of a space X into X is endowed with the compact-open topology. If X is locally compact and separable, then the space X^X is completely metrizable and separable [7, Theorem 3, p. 94].

A map $f: X \rightarrow Y$ is called k -dimensional if $\dim f = k$, where $\dim f = \sup\{\dim f^{-1}(y): y \in Y\}$.

A map $f: X \rightarrow Y$ is said to be locally closed if each point $x \in X$ has a closed neighborhood V such that the partial map $f|_V$ is closed. Notice that every mapping of a locally compact space is locally closed.

We use the following notation: $S(X) = \{f \in X^X: f \text{ is onto}\}$, $F_k(X) = \{f \in X^X: \dim f \leq k\}$, $F(X) = \{f \in X^X: \dim f < \infty\}$, $O(X) = \{f \in X^X: f \text{ is open}\}$, $OF_k(X) = O(X) \cap F_k(X)$, $OF(X) = O(X) \cap F(X)$, $OS(X) = O(X) \cap S(X)$, $OSF_k(X) = OS(X) \cap F_k(X)$, $OSF(X) = OS(X) \cap F(X)$, $L(X) = \{f \in X^X: f \text{ is locally closed}\}$, $LF_k(X) = L(X) \cap F_k(X)$, $LF(X) = L(X) \cap F(X)$.

Recall the concept of a local dimension, $\text{locdim } X$, of a space X . Let

$$L_n = \{x \in X: \text{there exists a neighborhood } U \text{ of } x \text{ such that } \dim U \leq n\}.$$

We say $\text{locdim } X \leq n$ provided $X = L_n$. By the Dowker–Nagami theorem $\text{locdim } X = \dim X$ if X is paracompact (see [1, Theorem 21, p. 284]). Clearly, all the sets L_n are open in X and $\dim L_n = \text{locdim } L_n \leq n$.

Proposition 1. *Let U be an arbitrary open nonempty subset of a space X .*

- (i) *If X is homogeneous with respect to $LF_0(X)$ and $\dim X = n < \infty$, then $\dim U = n$.*
- (ii) *If X is homogeneous with respect to $LF_k(X)$ and $\dim X = \infty$, then $\dim U = \infty$.*
- (iii) *If X is homogeneous with respect to $LF(X)$ and X is not strongly countable-dimensional, then $\dim U = \infty$; if, in addition, X is completely metrizable, then U is not strongly countable-dimensional.*

Proof. Let

$$M = \begin{cases} LF_0(X) & \text{in case (i) ,} \\ LF_k(X) & \text{in case (ii) ,} \\ LF(X) & \text{in case (iii) .} \end{cases}$$

Fix a point $x \in U$. For every $y \in X$ there exist a surjection $f \in M$ sending y to x and a closed neighborhood V of y such that the partial mapping $f|_V$ is closed and $f(V) \subset U$. By the Theorem on Dimension-Lowering Mappings [1, Corollary 1, p. 452],

$$\dim V \leq \dim f(V) + \dim f|_V \leq \dim U + \dim f.$$

In case (i) suppose $\dim U \leq n - 1$. Then by the above inequality $\dim V \leq n - 1$, since $\dim f = 0$. So, $X = L_{n-1}$, i.e. $\dim X = \text{locdim } X \leq n - 1$,

a contradiction. Similarly, if we suppose $\dim U = m < \infty$, then in case (ii) $X = L_{m+k}$, i.e. $\dim X = \text{locdim } X \leq m + k < \infty$, whereas in case (iii) $X = \bigcup \{L_n : n \geq m\}$; in both the cases we get a contradiction (notice that L_n 's are F_σ -sets).

Assume the space X in (iii) is completely metrizable. Suppose U is strongly countable-dimensional. Let $U = \bigcup \{F_n : n = 1, 2, \dots\}$, where F_n is closed in U and $\dim F_n < \infty$ for each n . By the Baire Category Theorem $\text{int } F_n \neq \emptyset$ for some n . We can assume that the point x belongs to $\text{int } F_n$ and replace the set U by $\text{int } F_n$. Just like in the paragraph above we obtain $X = \bigcup \{L_n : n \geq m\}$.

Corollary 1. *If a completely metrizable, strongly countable-dimensional space X is homogeneous with respect to $LF_k(X)$, then $\dim X < \infty$.*

Proof. Suppose $\dim X = \infty$. Let $X = \bigcup \{X_i : i = 1, 2, \dots\}$, where X_i is closed in X and $\dim X_i < \infty$ for every i . There exists i such that $\text{int } X_i \neq \emptyset$. In view of Proposition 1(ii) $\dim \text{int } X_i = \infty$, a contradiction.

A part of Proposition 2 (concerning the set $O(X)$ for a compact space X) appeared in [2, Proposition 2.1, p. 582] with a wrong proof. A correction had been suggested by Aarno Hohti.

Proposition 2. *If X is a locally compact, separable space, then the sets $O(X)$, $OS(X)$ and $H(X)$ are $F_{\sigma\delta}$ -sets in X^X .*

Proof. Let $\{B_1, B_2, \dots\}$ be a countable basis of open subsets of X such that $\text{cl } B_n$ is compact for every n . For each n the set B_n can be represented as the union $B_n = \bigcup \{C_n^k : k = 1, 2, \dots\}$ of compact sets C_n^k such that $C_n^k \subset \text{int } C_n^{k+1}$ for $k = 1, 2, \dots$. Let $Z = \{(m, n) : \text{cl } B_m \subset B_n\}$. For every $(m, n) \in Z$ and every rational number $r > 0$ put

$$F_{m,n,k,r} = \{f \in X^X : B(f(B_m), r) \subset f(C_n^k)\},$$

where $B(f(B_m), r)$ denotes the ball of radius r around $f(B_m)$. Since the convergence in X^X means the uniform convergence on each compact subset of X , it is easy to check that the sets $F_{m,n,k,r}$ are closed in X^X . Observe that

$$O(X) = \bigcap_{(m,n) \in Z} \bigcup_k \bigcup_r F_{m,n,k,r},$$

so $O(X)$ is an $F_{\sigma\delta}$ -subset of X^X . Furthermore, represent the space X as the union $X = \bigcup \{C_i : i = 1, 2, \dots\}$, where the sets C_i are compact and $C_i \subset \text{int } C_{i+1}$ for $i = 1, 2, \dots$, and define $A_{ij} = \{f \in X^X : C_i \subset f(C_j)\}$. The sets A_{ij} are closed in X^X and we have $OS(X) = O(X) \cap \bigcap_i \bigcup_j A_{ij}$. Thus $OS(X)$ is an $F_{\sigma\delta}$ -set in X^X .

Finally, $H(X) = OS(X) \cap \{f \in X^X : f \text{ is one-to-one}\}$. Notice that the set $\{f \in X^X : f \text{ is one-to-one}\}$ is a G_δ -subset of X^X . Indeed, the reasoning in the proof of [7, Theorem 1, p. 91] is suitable for locally compact spaces. Hence $H(X)$ is an $F_{\sigma\delta}$ -set in X^X .

We have also an easy generalization of a known result concerning $F_k(X)$ for a compact space X [7, Theorem 5, p. 109].

Proposition 3. *If X is a locally compact, separable space, then $F_k(X)$ is a G_δ -set and $F(X)$ is a $G_{\delta\sigma}$ -set in X^X .*

Proof. We refer to [7, §45, IV] for a definition and basic properties of the n -dimensional degree $d_n(X)$.

Represent X in the form

(a) $X = \bigcup \{C_n : n = 1, 2, \dots\}$, where C_n is compact and $C_n \subset \text{int } C_{n+1}$ for every n . Put

$$D_{n,\varepsilon} = \{f \in X^X : \text{there exists } y \in f(C_n) \text{ such that } d_{k+1}(f^{-1}(y) \cap C_n) \geq \varepsilon\}.$$

We show that

(b) $D_{n,\varepsilon}$ is closed in X^X .

In fact, let $f_m \in D_{n,\varepsilon}$ and $\lim_m f_m = f$ in the sense of [7, (7), p. 94]. There exist points $y_m \in f(C_n)$ such that

(c) $d_{k+1}(f_m^{-1}(y_m) \cap C_n) \geq \varepsilon$. We may assume $\lim_m y_m = y_0$. We claim that $d_{k+1}(f^{-1}(y_0) \cap C_n) \geq \varepsilon$. For suppose the inverse inequality holds. Then there exist open sets $G_0, \dots, G_t$ in C_n satisfying conditions $f^{-1}(y_0) \cap C_n \subset G_0 \cup \dots \cup G_t$, $\text{diam } G_i < \varepsilon$ and $G_{i_0} \cap \dots \cap G_{i_{k+1}} = \emptyset$ for $i_0 < \dots < i_{k+1} \leq t$. If there is a subsequence $\{m_s\}$, $s = 1, 2, \dots$, such that

$$A_{m_s} = f_{m_s}^{-1}(y_{m_s}) \cap C_n - (G_0 \cup \dots \cup G_t) \neq \emptyset \quad \text{for each } s,$$

then choose $x_{m_s} \in A_{m_s}$. Assume, without loss of generality, that the sequence $\{x_{m_s}\}$ tends to a point $x_0 \in C_n - (G_0 \cup \dots \cup G_t)$. By [7, (8), p. 94] $y_0 = \lim_s y_{m_s} = f(x_0)$, whence $x_0 \in f^{-1}(y_0) \cap C_n \subset G_0 \cup \dots \cup G_t$, a contradiction. Thus $f_{m_s}^{-1}(y_{m_s}) \cap C_n \subset G_0 \cup \dots \cup G_t$ for sufficiently large m ; this means that $d_{k+1}(f_m^{-1}(y_m) \cap C_n) < \varepsilon$ for such m . This contradicts (c) and the proof of (b) is complete.

The set $E_{n,\varepsilon} = X^X - D_{n,\varepsilon}$ is open. Put $E_n = \bigcap \{E_{n,1/i} : i = 1, 2, \dots\}$. We have

$$E_n = \{f \in X^X : \dim(f^{-1}(y) \cap C_n) \leq k \text{ for every } y \in f(C_n)\}.$$

Now, $F_k(X) = \bigcap \{E_n : n = 1, 2, \dots\}$ by (a) and the Sum Theorem for Dimension, whereas $F(X) = \bigcup \{F_k(X) : k = 1, 2, \dots\}$.

The proof is complete.

Corollary 2. *If X is a locally compact, separable space, then the sets $OF_k(X)$ and $OSF_k(X)$ are $F_{\sigma\delta}$ -sets, whereas the sets $OF(X)$ and $OSF(X)$ are $F_{\sigma\delta\sigma}$ -sets in X^X .*

Proposition 4. *The sets $L(X)$, $LF(X)$ and $LF_0(X)$ are closed under composition.*

Proof. Let $f, g \in L(X)$ and $x \in X$. Consider closed neighborhoods V of x and W of $f(x)$ such that the mappings $f|V$ and $g|W$ are closed. By the continuity of f there exists a closed neighborhood U of x such that $U \subset V$ and $f(U) \subset W$. Then the mapping $gf|U$ is closed, i.e. $gf \in L(X)$.

Now, assume additionally that the mappings f, g belong to $F(X)$. We will show that for every $y \in X$, $\text{locdim}(gf)^{-1}(y) \leq \dim f + \dim g$, whence

$$(a) \quad \dim(gf)^{-1}(y) \leq \dim f + \dim g.$$

Indeed, let $x \in (gf)^{-1}(y)$ and $U' = U \cap (gf)^{-1}(y)$, where U is the closed neighborhood of x described in the preceding paragraph. The Theorem on Dimension-Lowering Mappings [1, Corollary 1, p. 452] yields that

$$\begin{aligned} \dim U' &\leq \dim f|U' + \dim f(U') \leq \dim f + \dim f(U') \\ &\leq \dim f + \dim gf(U') + \dim g|f(U') \leq \dim f + \dim g. \end{aligned}$$

If $f, g \in LF_0(X)$, then $gf \in LF_0(X)$ by (a).

The proof is complete.

Recall that the quasi-interior of $A \subset X$ is the set

$$A^* = \bigcup \{U : U \text{ is open in } X \text{ and } U - A \text{ is of the first category}\}.$$

For a fixed point $a \in X$ denote by T_a the evaluation mapping $T_a: X^X \rightarrow X$, $T_a(f) = f(a)$.

Two versions of the Effros theorem [4, Theorem 2.1, p. 39] are proven in [2, Proposition 3.1 and Theorem 5.9]. The space X is assumed there to be compact, however one can observe that the same proofs work if X is locally compact and separable. Besides, in [2, Theorem 5.9] one can consider as M any Borel subset of $OS(X)$ which is closed under composition. So, in view of Corollary 2 and Proposition 4 the following proposition holds for $M = OSF_0(X)$ or $M = OSF(X)$.

Proposition 5. *Let X be a locally compact, separable space which is homogeneous with respect to a Borel subset M of $OS(X)$ such that if $f, g \in M$, then $gf \in M$. Then for each pair of points $a, b \in X$ there exists $f \in M$ such that $f(a) = b \in (T_a(H))^*$ for each H open in M with $f \in H$.*

Let n be a positive integer or $n = \infty$. An n -dimensional compact space is called an n -dimensional Cantor manifold if

- (i) no closed subset of X of dimension $\leq n - 2$ separates X , provided that $n < \infty$ or
- (ii) no finite-dimensional closed subset separates X , provided $n = \infty$.

Proposition 6 [6, Theorem VI 8, p. 94]. *If $1 \leq n < \infty$, then each n -dimensional compact space contains an n -dimensional Cantor manifold.*

Proposition 7 [8, Theorem, p. 245]. *Each ∞ -dimensional compact space either contains closed subsets of arbitrary finite dimension or it contains an ∞ -dimensional Cantor manifold.*

Lemma 1.

- (i) If f is a 0-dimensional mapping of an n -dimensional Cantor manifold X and $1 \leq n < \infty$, then no closed subset of dimension $\leq n-2$ separates $f(X)$ and $\dim f(X) \geq n$.
- (ii) If f is a finite-dimensional mapping of an ∞ -dimensional Cantor manifold X , then $f(X)$ is an ∞ -dimensional Cantor manifold.

Proof. In case (i) $\dim f(X) \geq n$ by the Theorem on Dimension-Lowering Mappings. Suppose a closed subset A separates $f(X)$ and $\dim A \leq n-2$. Then $f^{-1}(A)$ is a closed subset separating X with $\dim f^{-1}(A) \leq n-2$ by the theorem, a contradiction. The proof of (ii) is similar.

Lemma 2. Let X be a connected, locally compact space.

- (i) If X is homogeneous with respect to $OF_0(X)$ and contains an n -dimensional compact subset, $1 \leq n < \infty$, then no closed subset of dimension $\leq n-2$ separates X .
- (ii) If X is homogeneous with respect to $OF(X)$ and contains an ∞ -dimensional Cantor manifold, then no finite-dimensional closed subset separates X .

Proof. We will prove (i) and (ii) simultaneously. Obviously, we can assume that X is homogeneous with respect to $OSF_0(X)$ and $OSF(X)$, respectively. Represent X as the union $X = \bigcup \{C_m : m = 1, 2, \dots\}$, where C_m is compact and $C_m \subset \text{int } C_{m+1}$ for every m . If ρ is a metric in X , then the space X^X is metrizable by the metric $\hat{\rho}$ defined in the following manner: for $f, g \in X^X$ put $\rho_m(f, g) = \min\{1/m, \sup\{\rho(f(x), g(x)) : x \in C_m\}\}$, $\hat{\rho}(f, g) = \sup\{\rho_m(f, g) : m = 1, 2, \dots\}$ ([3, 8.5, p.272]).

Assume $A \subset X$ is an n -dimensional Cantor manifold in case (i) (Proposition 6) or an ∞ -dimensional Cantor manifold in case (ii). Suppose a closed subset B separates X and $\dim B \leq n-2$ in case (i) or $\dim B < \infty$ in case (ii). The space X being connected, the set B is nonempty. We can assume that $X-B = K \cup L$, where K and L are open, disjoint and $\text{cl } K = K \cup B$, $\text{cl } L = L \cup B$. Choose $a \in A$, $b \in B$ and $f \in OSF_0(X)$ in case (i) or $f \in OSF(X)$ in case (ii) with $f(a) = b$. By Lemma 1 $\dim f(A) \geq n$ in case (i) or $\dim f(A) = \infty$ in case (ii). Hence, $f(A) - B \neq \emptyset$, say $f(A) \cap K \neq \emptyset$. Take $c \in f(A) \cap K$ and put $\alpha = \rho(c, \text{cl } L) > 0$. Let m be such that $A \subset C_m$. Denote by $N_\varepsilon(f)$ the ε -ball about f in $OSF_0(X)$ (case (i)) or in $OSF(X)$ (case (ii)), where $0 < \varepsilon < \min\{1/m, \alpha\}$. By Proposition 5, $b \in (T_a(N_\varepsilon(f)))^*$. The open set $(T_a(N(f)))^*$ intersects L . We infer from the definition of the quasi-interior that there exists a point $d \in (T_a(N_\varepsilon(f)))^* \cap L$ such that $d \in T_a(N_\varepsilon(f))$. Thus, there is a mapping $g \in N_\varepsilon(f)$ such that $g(a) = d$. It follows from the definition of the metric $\hat{\rho}$ that if $c' \in A$ and $f(c') = c$, then $\rho(g(c'), c) < \varepsilon < \alpha$. So, $g(A) \cap K \neq \emptyset \neq g(A) \cap L$. Therefore the closed set $g(A) \cap B$ separates $g(A)$ and, besides, its dimension is less than $n-1$ in case (i) or is finite in case (ii). This contradicts Lemma 1 and completes the proof.

Theorem 1. *Let X be a locally compact, connected space.*

- (i) *If $1 \leq \dim X = n < \infty$ and X is homogeneous with respect to $OF_0(X)$, then no subset of dimension $\leq n - 2$ separates X .*
- (ii) *If $\dim X = \infty$ and X is homogeneous with respect to $OF_0(X)$, then no finite-dimensional subset separates X .*

Proof. It is known that each separator of a hereditarily normal space contains a closed separator. So, part (i) follows from Lemma 2(i). Part (ii) is a consequence of Proposition 7 and Lemma 2.

Proposition 8 [1, p. 534]. *A closed subset of an A -w.i.d. space is also A -w.i.d. A countable union of A -w.i.d. spaces is A -w.i.d..*

Proposition 9. *If an S -w.i.d. space X is homogeneous with respect to $LF_k(X)$ and $\dim X = \infty$, then X is compact.*

Proof. Suppose X is not compact. There exists a discrete countable subset $\{x_1, x_2, \dots\} \subset X$. Since the sets L_n are open in X and $\dim L_n \leq n$ (see the paragraph preceding Proposition 1), it follows from Proposition 1 that $\bigcup \{L_n : n = 1, 2, \dots\} = \emptyset$. However, the sequence $\{L_n\}$ converges in the sense of [1, Lemma 2, p. 539]. This means that there is a number s such that if $t \geq s$, then $x_t \in \bigcup \{L_n : n = 1, 2, \dots\}$, a contradiction.

The following theorem is a corollary to Theorem 1 and Proposition 9.

Theorem 2. *If X is an ∞ -dimensional, S -w.i.d., connected, locally compact space which is homogeneous with respect to $OF_0(X)$ then, X is an ∞ -dimensional Cantor manifold.*

A continuum X is said to be an s.i.d. Cantor manifold if no closed w.i.d. subset separates X .

The following proposition is due to Skljarenko (see [1, Theorem 32, p. 550]).

Proposition 10. *Each s.i.d. compact space contains an s.i.d. Cantor manifold.*

Theorem 3. *If X is a homogeneous connected, locally compact A -s.i.d. space, then no A -w.i.d. subset separates X .*

Proof. We have $X = \bigcup \{C_n : n = 1, 2, \dots\}$, where C_n is compact and $C_n \subset C_{n+1}$ for each n . It follows from Proposition 8 that C_n is s.i.d. for some n . Thus, by Proposition 10, C_n contains an s.i.d. Cantor manifold A . The proof now proceeds as that of Lemma 2.

Remarks.

1. It is seen from the proof of Lemma 2 that if one knew that an image of an s.i.d. compact space under a finite-dimensional mapping is again s.i.d., Theorem 3 would be valid under a weaker assumption of homogeneity with respect to $OF(X)$. The problem can be reduced to the question of whether 0-dimensional mappings preserve strong infinite dimension of continua.

2. A problem posed in [5, p. 9] is whether there exists a countable-dimensional, infinite-dimensional homogeneous continuum. It follows from Theorem 1 that such a continuum must be of transfinite inductive dimension $> \omega_0$.

3. As pointed out by J. R. Prajs there are examples of 2-dimensional continua which are homogeneous with respect to 0-dimensional mappings but are separated by points [not published]. Nevertheless, one can ask if Theorem 1 holds for spaces homogeneous with respect to larger (Borel) classes of mappings, e.g. for confluent 0-dimensional mappings.

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