ON THE TYPE OF WIENER-HOPF C^* -ALGEBRAS

ALBERT JEU-LIANG SHEU

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ABSTRACT. In this paper, we show that if the positive cone P of a locally compact group G does not satisfy a regularity condition then the corresponding Wiener-Hopf C^* -algebra $\mathscr{W}(P)$ is not of type I while the converse does not hold, and that if $C^*(G)$ is not of type I then neither is $\mathscr{W}(P)$. Thus a conjecture and a question, both proposed by P. Muhly and J. Renault in their important systematic treatment of general Wiener-Hopf C^* -algebras using groupoid C^* -algebras, are settled.

INTRODUCTION

In [M-R], Muhly and Renault initiated a systematic study of the C^* -algebras $\mathscr{W}(P)$ of Wiener-Hopf operators on normal subsemigroups P of locally compact groups G , using a completely new approach; namely, the method of groupoid C^* -algebras [C, R1]. There they made a conjecture that if P does not satisfy a certain regularity condition (see below) then $\mathscr{W}(P)$ does not contain the algebra $\mathscr{K}$ of compact operators. In the first section of this paper, we prove that conjecture by showing that for such P , $\mathscr{W}(P)$ is a non-type-I C^* -algebra and contains no nontrivial compact operators. This result in particular implies a result of [D]. On the other hand, it is also asked in [M-R] whether the regularity condition of P implies that $\mathscr{W}(P)$ is of type I (i.e., the converse of the conjecture). In the second section of this paper, we give a negative answer to this question. In fact, we show that, whether P satisfies the regularity condition or not, $\mathscr{W}(P)$ is never of type I unless $C^*(G)$ is of type I.

1.

Let G be a second countable, locally compact group with identity e and left Haar measure λ fixed and P be a closed normal subsemigroup of G containing e such that P is the closure of its interior $\text{Int}(P)$ (and hence of positive measure). We also assume that P generates G and $\{e\} = P \cap P^{-1}$. By definition,

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the Wiener–Hopf C^* -algebra $\mathscr{W}(P)$ (of the pair (G, P)) is the C^* -algebra generated by the Wiener–Hopf operators $\mathscr{W}(f)$ on $L^2(P, \lambda)$, $f \in C_c(G)$, where

$$\mathscr{W}(f)\xi(t) := \int_G f(s)\xi(ts)\chi_P(ts) d\lambda(s)$$

for $\xi \in L^2(P, \lambda)$. Let $\mathscr{A}$ be the (commutative) C^* -subalgebra of $C_b(G)$ generated by $\chi_P * f$ with $f \in L^1(G)$, where

$$\chi_P * f(t) = \int_G \chi_P(ts)f(s^{-1}) d\lambda(s),$$

and let Y be the maximal ideal space of $\mathscr{A}$ (i.e., $\mathscr{A} = C_0(Y)$). It is easy to see that $\mathscr{A}$ is invariant under the G -action by (right) translation and hence we have a corresponding G -action on Y . Clearly G is imbedded in Y through evaluation and so we may define X to be the closure of P in Y . By [M-R, Lemma 3.3], X is compact. Now we define $\mathfrak{G}$ to be the transformation group groupoid $Y \times G$ reduced to X with the Haar system inherited from that of $Y \times G$ defined by $\{\delta_y \times \lambda\}_{y \in Y}$ [M-R]. Then one of the main results of [M-R] says that the (reduced) groupoid C^* -algebra $C^*(\mathfrak{G})$ is isomorphic to the Wiener–Hopf C^* -algebra $\mathscr{W}(P)$. More precisely, they proved that the induced representation $\text{Ind}(\delta_e)$ of $C^*(\mathfrak{G})$ on $L^2(\mathfrak{G}, \lambda^e)$ is faithful and its image coincides with $\mathscr{W}(P)$ if we identify $L^2(\mathfrak{G}, \lambda^e)$ and $L^2(P, \lambda)$ suitably [M-R]. Recall that under this identification, $\text{Ind}(\delta_e)$ becomes the representation π on $L^2(P, \lambda)$ defined by

$$\pi(f)\xi(t) = \int_G f(t, s)\xi(ts)\chi_P(ts) d\lambda(s)$$

for f in $C_c(\mathfrak{G})$ and ξ in $L^2(P, \lambda)$.

Recall that X is called a regular compactification of P if P is open in X and the embedding of P in X is a homeomorphism from P to the image. If X is not a regular compactification of P , it is conjectured in [M-R, 3.7.3] that $\mathscr{W}(P)$ does not contain the algebra $\mathscr{K}$ of compact operators. One of our main results is the proof of this conjecture.

For the convenience of discussion, we list three conditions on P (which will be shown to be equivalent in Lemma 2).

- (1) X is not a regular compactification of P .
- (2) There is a sequence $p_n \in P$ diverging to ∞ in G but converging to some $p \in P$ in X .
- (3) For any $p \in \text{Int}(P)$, there is a sequence $p_n \in \text{Int}(P)$ diverging to ∞ in G but converging to p in X .

Lemma 1. *If $p_n \in G$ converges to a $p \in \text{Int}(P)$ in Y , then $p_n \in \text{Int}(P)$ for n sufficiently large.*

Proof. Now clearly $p_n p^{-1}$ converges to e in Y since Y is a G -space. Given any neighborhood V of e in G and any nonnegative $f \in C_c(V)$ with $f(e) > 0$, it is easy to see that $\text{supp}(\chi_P * f) \subseteq PV$ and $\chi_P * f(e) > 0$ (since P is the

closure of $\text{Int}(P)$). Now since $\chi *_P f(p_n p^{-1})$ converges to $\chi *_P f(e)$, we have $p_n p^{-1} \in PV$ for n large. Choosing V sufficiently small, we have $Vp \subseteq \text{Int}(P)$ since $p \in \text{Int}(P)$, and hence $p_n \in PVp \subseteq P \text{Int}(P) \subseteq \text{Int}(P)$. Q.E.D.

Lemma 2. *Conditions (1), (2), and (3) are all equivalent.*

Proof. Condition (1) implies that either (i) there is a sequence $\tilde{q}_n$ in $X \setminus P$ converging to some $q \in P$ in X (or equivalently in Y), or (ii) the embedding of P in X is not a homeomorphism from P to its image in X . First we assume (i), and then by considering a sequence K_n of compact subsets of G such that $G = \bigcup_n K_n$ and $K_n \subseteq \text{Int}(K_{n+1})$, we can easily get a sequence $q_n \in P \setminus K_n$ (and close to $\tilde{q}_n$) converging to q in X such that q_n diverges in G to ∞ . Next we assume (ii), and then there is a sequence $p_n \in P$ converging to some $p \in P$ in X but not in G . Since, by elementary facts about compact Hausdorff spaces, the embedding restricted to any compact subset of P is a homeomorphism, it is easy to see that p_n has to diverge to ∞ in G (note that P is closed in G). So, in both cases, condition (2) holds and hence (1) implies (2).

Now assuming condition (2), we get, say, a sequence $q_n \in P$ converging to some $q \in P$ in X such that q_n diverges in G to ∞ . Since G acts on Y by homeomorphisms, we have $q_n q^{-1} \in G$ converges to e in Y . Now for any $p \in \text{Int}(P)$, $q_n q^{-1} p \in G$ converges to p in Y and diverges to ∞ in G . So all we need to show is that $q_n q^{-1} p$ is in $\text{Int}(P)$ for n large, but this comes from Lemma 1. Thus (2) implies (3).

It is obvious that condition (3) implies (1) since $\text{Int}(P)$ is not empty. Q.E.D.

Theorem 1. *With the notations as above, if X is not a regular compactification of P , then $\mathcal{W}(P)$ is not of type I and contains no nontrivial compact operators.*

Proof. First let us point out that the induced representation $\text{Ind}(\delta_e)$ is in fact irreducible. This can be seen through the results of [R2]. In fact, it is easy to check that $\text{Ind}(\delta_e)$ can be disintegrated into the integral of the representation $(m, \mathcal{H})$ of the dynamical system $(\mathfrak{G}, \mathfrak{G}, \mathbb{C})$ (c.f. [R2, Theorem 4.1(i)]), where m is the transverse measure of $\mathfrak{G}$ determined by the measure $\mu = \lambda|_P$ on $\mathfrak{G}^{(0)} = X$ (c.f. [R2, Proposition 2.2]) and $\mathcal{H}$ is the trivial one-dimensional Hilbert bundle over X . Clearly $(m, \mathcal{H})$ is irreducible and hence, by [R2, Theorem 4.1(ii)], $\text{Ind}(\delta_e)$ is irreducible. Thus by [Ar, Corollary 2 of Theorem 1.4.2], the image of $\text{Ind}(\delta_e)$ either contains $\mathcal{K}$ or intersects with $\mathcal{K}$ on $\{0\}$. We shall prove the latter to be true.

For any $(p_0, g_0) \in \mathfrak{G} \subseteq X \times G$ such that $p_0 \in \text{Int}(P)$ and $p_0 g_0 \in \text{Int}(P)$ (note that such points are dense in $\mathfrak{G}$), we can find a sequence $p_n \in \text{Int}(P)$ converging to $p_0 \in \text{Int}(P)$ in X , by Lemma 2. We claim that there is a neighborhood V of e in G such that $p_n V \subseteq \text{Int}(P) \subseteq X$ for all $n \geq 0$, since otherwise, there are a subsequence $p_{n(k)}$ of p_n and a sequence $g_{n(k)} \in G$ converging to e in G such that $p_{n(k)} g_{n(k)} \notin \text{Int}(P)$ for all k , but clearly $p_{n(k)} g_{n(k)}$ converges to $p_0 e = p_0 \in \text{Int}(P)$ in X which contradicts Lemma 1. Let U be a relative

compact neighborhood of e in G with the closure $\overline{U} \subseteq V$ (so $0 < \lambda(U) < \infty$). Note that since p_n diverges to ∞ in G , by taking a subsequence of p_n if necessary, we may assume that $p_n U$'s are all disjoint.

Now for any $f \in C_c(\mathfrak{G})$, we have $f(p_n p_0^{-1} t, t^{-1} s)$ converging uniformly to $f(t, t^{-1} s)$ on $(t, s) \in (p_0 U) \times (p_0 U)$, since the map sending (p, t, s) to $(p p_0^{-1} t, t^{-1} s)$ is a continuous map from the compact $\{p_n | n \geq 0\} \times (p_0 \overline{U}) \times (p_0 \overline{U}) \subseteq X \times G \times G$ to $\mathfrak{G}$.

Let $\pi_n(f)$ be the operator $\pi(f)$ compressed to $L^2(p_n U, \lambda)$ for all $n \geq 0$. Then

$$\pi_n(f)\xi(t) = \int f_n(t, t^{-1}s)\xi(s) d\lambda(s)$$

for $\xi \in L^2(p_n U, \lambda)$ and $t \in p_n U$ (it is understood that ξ is extended by 0 outside $p_n U$). Let τ_n be the unitary operator from $L^2(p_n U, \lambda)$ to $L^2(p_0 U, \lambda)$ defined by

$$(\tau_n \xi)(t) = \xi(p_n p_0^{-1} t)$$

for $t \in p_0 U$. Then the operator $\tau_n \pi_n(f) \tau_n^{-1}$ on $L^2(p_0 U, \lambda)$ satisfies

$$(\tau_n \pi_n(f) \tau_n^{-1} \xi)(t) = \int f(p_n p_0^{-1} t, t^{-1} s) \xi(s) d\lambda(s)$$

for $\xi \in L^2(p_0 U, \lambda)$ and $t \in p_0 U$. By the above uniform convergence, it is easy to see that $\|\tau_n \pi_n(f) \tau_n^{-1} - \pi_0(f)\|$ converges to 0 and hence $\|\pi_n(f)\|$ converges to $\|\pi_0(f)\|$ as n goes to infinity. Since the $p_n U$'s are all disjoint and hence the $L^2(p_n U, \lambda)$'s are orthogonal to one another, we have $\|\pi(f) - T\| \geq \|\pi_0(f)\|$ for any compact operator T on $L^2(P, \lambda)$.

Now let $W = p_0 U$ be as above and $\xi = \lambda(U)^{-1/2} \chi_W$ be a unit vector in $L^2(p_0 U, \lambda) \subseteq L^2(P, \lambda)$. If the rank one projection $T := \langle \cdot, \xi \rangle \xi$ is in $\mathscr{W}(P)$, then there is a sequence $\pi(f_m)$ with $f_m \in C_c(\mathfrak{G})$ such that $\lim \|\pi(f_m) - T\| = 0$ and hence $\lim \|\pi_0(f_m)\| = 1$. But then by the above result $\|\pi(f_m) - T\| \geq \|\pi_0(f_m)\| \geq 1/2$ for all large m , a contradiction. So we have proved that $T \notin \mathscr{W}(P)$, and hence $\mathscr{W}(P) \cap \mathscr{K} = \{0\}$ and $\mathscr{W}(P)$ is not of type I since $\pi = \text{Ind}(\delta_e)$ is irreducible. Q.E.D.

A consequence of Theorem 1 is the following result of [D].

Corollary. For $P = \{z \in \mathbf{Z}^n | \iota_\alpha(z) \geq 0\} \subseteq G = \mathbf{Z}^n$ with $n \geq 2$, the Wiener-Hopf C^* -algebra $\mathscr{W}(P)$ is non-type-I and contains no nontrivial compact operators, where $\iota_\alpha(z) := \sum_{i=1}^n z_i \alpha_i$ for any fixed $\mathbf{Q}$ -linearly independent real numbers α_i and for all $z \in \mathbf{Z}^n$.

Proof. Since G is discrete, $\mathscr{A}$ is the (commutative) C^* -subalgebra of $l^\infty(G)$ generated by the characteristic functions χ_{P+z} with $z \in \mathbf{Z}^n$. Since α_i 's are $\mathbf{Q}$ -linearly independent and $n \geq 2$, we have the closure of $\iota_\alpha(P)$ in $\mathbf{R}$ equal to $[0, \infty)$, and hence we can find a sequence $\{y(k)\}_{k \in \mathbf{N}} \subseteq P$ diverging to infinity such that $\lim \iota_\alpha(y(k)) = 0$. Now clearly $y \in P + z$ if and only if

$\iota_\alpha(y) \geq \iota_\alpha(z)$, so $0 \in P + z$ if and only if $y(k) \in P + z$ for k sufficiently large (since $\iota_\alpha(y(k)) \geq 0$ for all k). Thus $\lim \chi_{P+z}(y(k)) = \chi_{P+z}(0)$ for all z in $\mathbb{Z}^n$, and hence $y(k)$ converges to 0 in Y as characters of $\mathcal{A}$. Now by Lemma 2, X is not a regular compactification of P and so by Theorem 1, we get the statement. Q.E.D.

2.

In [M-R, 3.7.3], it is asked whether X being a regular compactification of P implies that $\mathcal{W}(P)$ is of type I. We shall give a negative answer to this question. The point is that unless (the reduced) $C^*(G)$ itself is of type I, the algebra $\mathcal{W}(P)$ cannot be of type I.

Theorem 2. *If $C^*(G)$ is not of type I, then $\mathcal{W}(P)$ is not of type I.*

Proof. We claim that for any compact $K \subseteq G$, there is a $g \in G$ such that $Pg \subseteq \bigcap_{k \in K} Pk$.

First we check this for finite K . Since P is assumed to be normal and generate G , we have $G = P^{-1}P$. So, for any k, h in G , $Pk \cap Ph$ is not empty and hence contains Pg for some $g \in G$ (in fact, for all $g \in Pk \cap Ph$). Thus, by iteration, any intersection of finitely many Pk 's, $k \in G$, is not empty and contains some Pg .

Now if we can show that for any $h \in G$ there is a (compact) neighborhood U with the property of the claim (i.e., $\bigcap_{u \in U} Pu$ contains some Pk), then we get the claim, since then we can cover K by finitely many such U 's and then, by the claim for the finite case, the intersection of the corresponding finitely many Pk 's will contain some Pg as desired. In fact, for any $h \in G$, we can find $k \in G$ such that $kh^{-1} \in \text{Int}(P)$ and so $kU^{-1} \subseteq \text{Int}(P)$ for a neighborhood U of h . Thus $k \in Pu$ for all $u \in U$ and hence $Pk \subseteq \bigcap_{u \in U} Pu$. So the claim is proved.

Now let $e = g_1, g_2, g_3, \dots$ be a sequence dense in G and with $C_n := \bigcap_{i=1}^n Pg_i \subseteq P$ nonempty for all n . Note that for any $g \in G$, since Pg contains the open set $\text{Int}(P)g$, we have Pg containing some g_i and hence Pg_i , and so $C_n \subseteq Pg$ for all $n \geq i$. Thus, using the claim, we get that for any compact $K \subseteq G$, there is an i such that $C_n \subseteq \bigcap_{k \in K} Pk$ for all $n \geq i$.

Let $p_n \in C_n$; then by the above result, it is easy to check that

$$\lim \chi_P * f(p_n) = \int_G \tilde{f}(s) d\lambda(s)$$

and similarly for any $g \in G$

$$\lim \chi_P * f(p_n g) = \int_G (\tau_g \tilde{f})(s) d\lambda(s) = \int_G \tilde{f}(s) d\lambda(s)$$

for all $f \in C_c(G)$, where $\tilde{f}(s) = f(s^{-1})$ and $\tau_g f(s) := f(g^{-1}s)$. So p_n converges to a G -invariant limit character q in X .

Now by the general theory of groupoid C^* -algebra developed in [R1], the invariant closed subspace $\{q\}$ gives rise to a closed subgroupoid $\mathfrak{G}_q = \{q\} \times G$

of $\mathfrak{G}$ whose groupoid C^* -algebra $C^*(\mathfrak{G}_q) \cong C^*(G)$ is a quotient of $C^*(\mathfrak{G})$. So $C^*(\mathfrak{G})$ cannot be of type I since $C^*(G)$ is not. Q.E.D.

Now we use Theorem 2 to show that regularity of (X, P) need not imply $\mathscr{W}(P)$ being of type I.

Consider the discrete Heisenberg group H ($= \mathbb{Z}^3$ as sets) whose group C^* -algebra has been extensively studied [An-Pas]. Recall that the group operation is defined by $(i, j, k) \cdot (a, b, c) = (a+i, b+ci+j, c+k)$ for a, b, c, i, j, k in $\mathbb{Z}$. Let

$$P = \{(a, b, c) | a, c > 0, a, b, c \in \mathbb{Z}\} \cup \{(0, 0, 0)\}.$$

Then one can check that P satisfies all the requirements (i.e., P is a normal subsemigroup generating H with $P \cap P^{-1} = \{e\}$). Since H is discrete, $\mathscr{A}$ is the commutative C^* -algebra generated by $\chi_{P \cdot z}$, $z \in \mathbb{Z}^3$, in $l^\infty(\mathbb{Z}^3)$. Clearly, $\chi_{P \cdot (0, 1, 0)}(0, 0, 0) = 0$ while $\chi_{P \cdot (0, 1, 0)}(a, b, c) = 1$ for all $(a, b, c) \in P \setminus \{(0, 0, 0)\}$. Thus $(0, 0, 0)$ is an isolated point in X and hence by Lemma 2, X is a regular compactification of P . But by Theorem 2, $\mathscr{W}(P)$ is not of type I since $C^*(H)$ is well known to be non-type-I. So this example gives a negative answer to the question of Muhly and Renault, and one may then ask, instead, whether the regularity of (X, P) and the condition that $C^*(G)$ is of type I imply that $\mathscr{W}(P)$ is also of type I. But the answer to this substitute question is still negative. In fact, the algebras $\mathscr{W}(P_{\alpha, \beta})$ (which have been studied closely in [Par]) provide such counterexamples when one of α and β is irrational, where $P_{\alpha, \beta}$ is the cone $\{(m, n) \in \mathbb{Z}^2 | -\alpha m + n \geq 0 \text{ and } -\beta m + n \leq 0\}$ (with $0 < \alpha < \beta$) in $G = \mathbb{Z}^2$ whose $C^*(G) = C(\mathbb{T}^2)$ is of type I. It can be checked that $\mathscr{W}(P_{\alpha, \beta})$ is of type I if and only if both α and β are rational, but the corresponding compactification $X_{\alpha, \beta}$ of $P_{\alpha, \beta}$ is always regular for any real α and β (with $0 < \alpha < \beta$).

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