SMALL DEFORMATIONS OF A CLASS OF COMPACT NON-KÄHLER MANIFOLDS

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ABSTRACT. Small deformations of Kähler manifolds are Kähler too; we prove here that this is not true for balanced manifolds nor, more generally, for p -Kähler manifolds ($p > 1$), i.e., the property of being p -Kähler is not stable under small deformations.

1. INTRODUCTION

The aim of this paper is to prove that if $M = \{M_t\}$ is a complex analytic family of compact complex manifolds, and M_{t_0} is p -Kähler ($p > 1$), then, even if $|t - t_0|$ is small, M_t may be a non p -Kähler manifold; i.e., the property of being p -Kähler is not stable under small deformations (see Theorem 2.3 for the definition of p -Kähler manifold).

It is well known that the situation is completely different in the case of Kähler manifolds [KS]. In what follows, we shall distinguish the case $p = \dim M_t - 1 := n - 1$ (§4) from the general case (§3): indeed, an $(n - 1)$ -Kähler manifold is nothing but a manifold equipped with a balanced metric, i.e., a hermitian metric h such that if ω is its Kähler form and δ is the formal adjoint of d in the metric h , it holds that $\delta\omega = 0$ (see [M] for a survey on balanced manifolds). The condition of being balanced (or co-Kähler) is, in a sense, dual to that of being Kähler, so it may be interesting to emphasize the different behaviour of small deformations of Kähler and balanced manifolds

2. PRELIMINARIES

(See also [HK].) Suppose V is a complex vector space of complex dimension n and let $\sigma_p := i^{p^2} 2^{-p}$.

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2.1. Definition. (1) An element of $\Lambda^{p,0}(V)$ that can be expressed as $v_1 \wedge \cdots \wedge v_p$ with $v_j \in V$ is called decomposable.

(2) $\psi \in \Lambda^{p,p}(V)$ is called strongly positive if ψ can be written as $\sum_j \sigma_p \zeta_j \wedge \bar{\zeta}_j$, with $\zeta_j \in \Lambda^{p,0}(V)$ decomposable.

(3) $\Omega \in \Lambda^{p,p}(V)$ is called weakly positive if $\Omega \wedge \psi$ is a positive volume for every strongly positive $(n-p, n-p)$ -vector.

Let SP^p and WP^p denote the cones of strongly positive and weakly positive (p, p) -vectors.

Let M be a connected complex manifold of dimension n ; we denote, as usual, by Ω^p the sheaf of germs of holomorphic p -forms and by E^p the sheaf of germs of C^∞ p -forms.

Moreover, let $E^{p,p}(M)_{\mathbf{R}}$ denote the Fréchet space of real (p, p) -forms, while $E'_{p,p}(M)_{\mathbf{R}}$ denotes its dual space of real currents of bidimension (p, p) .

2.2. Definition. $\Omega \in E^{p,p}(M)_{\mathbf{R}}$ is called *transverse* if $\forall x \in M$, Ω_x belongs to the interior of the cone $WP^p(T'_x M^*)$. $T \in E'_{p,p}(M)_{\mathbf{R}}$ is called *positive* if $T(\Omega) \geq 0$ for every transverse (p, p) -form Ω .

2.3. Theorem. Let M be a compact complex manifold of dimension n , and p an integer, $1 \leq p \leq n$. The following assertions are equivalent:

- (1) There exists on M a closed transverse (p, p) -form;
- (2) There are no nontrivial positive (p, p) -currents which are (p, p) -components of boundaries.

In this case, M is called a p -Kähler manifold.

Proof. See [AA]. $\square$

Remarks. (see [AA] and [AB]).

- (1) M is 1-Kähler if and only if M is Kähler.
- (2) M is $(n-1)$ -Kähler if and only if M is balanced.
- (3) If M is Kähler, then it is p -Kähler for every $p \leq n$.

More about p -Kähler manifolds can be found in [AA] and [AB]; we only recall here the definition of $\eta\beta_5$, a nontrivial 3-Kähler manifold that we shall use in the next section.

Let G be the following subgroup of $GL(4, \mathbf{C})$:

$$G := \left\{ X \in GL(4, \mathbf{C}) / X = \begin{pmatrix} 1 & z_1 & z_3 & z_5 \\ 0 & 1 & 0 & z_2 \\ 0 & 0 & 1 & z_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

and let Γ be the subgroup of G consisting of matrices with entries in $\mathbf{Z}[\sqrt{-1}]$. Γ is a discrete subgroup of G and G/Γ becomes a holomorphically parallelisable compact connected complex manifold of dimension 5, called $\eta\beta_5$. We can

identify the complex Lie group G with $(\mathbb{C}^5, *)$, where

$$(z_1, \dots, z_5) * (w_1, \dots, w_5) = (z_1 + w_1, \dots, z_4 + w_4, z_5 + w_5 + z_1 w_2 + z_3 w_4),$$

and we can choose a basis $\{\phi_j\}_{1 \leq j \leq 5}$ for $\Omega^1(\eta\beta_5)$ such that $d\phi_j = 0$ for $j = 1, \dots, 4$ and $d\phi_5 = -\phi_1 \wedge \phi_2 - \phi_3 \wedge \phi_4$. Call $\{\theta_j\}$ the dual basis.

2.5. Proposition. $\eta\beta_5$ is not Kähler nor 2-Kähler, it is p -Kähler for $p = 3, 4, 5$.
Proof. See [AB].

The theory of small deformations can be found in [K] or [KM, Chapter 4].

3. THE CASE $p < n - 1$

Take $M = \eta\beta_5$ and $\psi(t) := t\theta_1\bar{\phi}_1$, $t \in \mathbb{C}$.

$\forall t \in \mathbb{C}$, the vector $(0, 1)$ -form $\psi(t)$ satisfies the equation

$$\bar{\partial}\psi(t) - \frac{1}{2}[\psi(t), \psi(t)] = 0,$$

and therefore, by Kuranishi's theorem [K], $\psi(t)$ defines for $|t| < \varepsilon$ a locally complete complex analytic family of deformations $\{M_t\}$ of M .

By solving

$$\begin{cases} \bar{\partial}\zeta_j - \psi(t)\zeta_j = 0, \\ \zeta_j(0) = z_j, \end{cases} \quad j = 1, \dots, 5,$$

we get

$$\begin{cases} \zeta_1 = z_1 + t\bar{z}_1 \\ \zeta_j = z_j, \end{cases} \quad j = 2, \dots, 5,$$

so that, if

$$\phi_{j,t} = d\zeta_{j,t} \quad \text{for } j = 1, \dots, 4$$

and

$$\phi_{5,t} = d\zeta_{5,t} + \frac{t\bar{\zeta}_{1,t} - \zeta_{1,t}}{1 - |t|^2} d\zeta_{2,t} - \zeta_{3,t} d\zeta_{4,t},$$

$\{\phi_{j,t}, \bar{\phi}_{j,t}\}_{j=1,\dots,5}$ becomes a basis for $H^0(M_t, E^1)$.

3.1. Proposition. For $|t| < \varepsilon$, M_t is balanced, but it is not 3-Kähler if $t \neq 0$.

Proof. A 4-Kähler form for M_t is $\Omega = \sigma_4 \Sigma'_{|A|=4} \phi_A \wedge \bar{\phi}_A$ (as usual, Σ' denotes the sum over all increasing multiindices, $A = (\alpha_1, \dots, \alpha_p)$ and $\phi_A = (\phi_{\alpha_{1,t}} \wedge \dots \wedge \phi_{\alpha_{p,t}})$).

Let η be the $(2, 2)$ -component of $d(\phi_{1,t} \wedge \phi_{5,t} \wedge \bar{\phi}_{2,t})$, i.e.,

$$\eta = \frac{t}{1 - |t|^2} \phi_{1,t} \wedge \phi_{2,t} \wedge \bar{\phi}_{1,t} \wedge \bar{\phi}_{2,t},$$

and let T_η be the real smooth $(3, 3)$ -current defined by η , that is,

$$T_\eta(\alpha) := \int_M \eta \wedge \alpha.$$

Since, if $\alpha \in WP^3(M_t)$, $\eta \wedge \alpha$ is a nonnegative multiple of the volume of M_t , T_η is a positive $(3, 3)$ -current which is the $(3, 3)$ -component of a boundary and T_η is trivial only for $t = 0$. We can conclude by Theorem 2.3. $\square$

4. THE CASE $p = n - 1$

Let us choose the simplest example of a compact balanced non-Kähler manifold, the Iwasawa manifold I_3 . Nakamura [N, pp. 94–96] builds up, via Kuranishi's theorem, a locally complete complex analytic family of deformations M_t of $M_0 = I_3$, depending on six effective parameters t_{ij} .

For our purpose it suffices to choose $t_{ij} = 0$ for $(i, j) \neq (1, 2)$. Call $t_{12} := t$: we get on M_t coordinates $\{\zeta_{1,t}, \zeta_{2,t}, \zeta_{3,t}\}$ such that, if

$$\phi_{j,t} = d\zeta_{j,t} \quad \text{for } j = 1, 2 \quad \text{and} \quad \phi_{3,t} = d\zeta_{3,t} + (t\bar{\zeta}_{2,t} - \zeta_{1,t})d\zeta_{2,t},$$

then

$$\{\phi_{j,t}, \bar{\phi}_{j,t}\}_{j=1,2,3} \text{ becomes a basis for } H^0(M_t, E^1).$$

If Ω were a closed transverse $(2, 2)$ -form on M_t , by Stokes' theorem,

$$0 = \int_{M_t} d(\Omega \wedge \sigma_1 \bar{t} \phi_{3,t}) = \int_{M_t} \Omega \wedge \sigma_1 \bar{t} d\phi_{3,t} = - \int_{M_t} \Omega \wedge \sigma_1 |t|^2 \phi_{2,t} \wedge \bar{\phi}_{2,t},$$

but, for $t \neq 0$, the last term is negative.

We have therefore proved the following:

4.1. Proposition. *There exists a nonbalanced small deformation of I_3 .*

REFERENCES

- [AA] L. Alessandrini and M. Andreatta, *Closed transverse (p, p) -forms on compact complex manifolds*, *Compositio Math.* **61** (1987), 181–200; *Erratum*, *ibidem* **63** (1987).
- [AB] L. Alessandrini and G. Bassanelli, *Compact p -Kähler manifolds* (to appear).
- [HK] R. Harvey and A. W. Knapp, *Positive (p, p) -forms, Wirtinger's inequality and currents*, in *Proceedings Tulane Univ., Program on Value Distribution Theory in Complex Analysis and Related Topics in Differential Geometry 1972–73*, Dekker, New York, 1974, 43–62.
- [K] M. Kuranishi, *On the locally complete family of complex analytic structures*, *Ann. Math.* **75** (1962), 536–577.
- [KM] K. Kodaira and J. Morrow, *Complex Manifolds*, Holt, Rinehart and Winston, New York, 1971.
- [KS] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures, III. Stability theorems for complex structures*, *Ann. Math.* **71** (1960), 43–76.
- [M] M. L. Michelson, *On the existence of special metrics in complex geometry*, *Acta Math.* **143** (1983), 261–295.
- [N] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, *J. Differential Geom.* **10** (1975), 85–112.

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