A PROOF OF THE PÓLYA-WIMAN CONJECTURE

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. Let $f(z) = e^{-\alpha z^2} g(z)$ where $\alpha \geq 0$ and g is a real entire function of genus at most 1. It is shown that if f has only a finite number of nonreal zeros, then its derivatives, from a certain one onward, have only real zeros.

1. INTRODUCTION

This paper is concerned with the zeros of successive derivatives of entire functions. In order to state concisely the background of our results as well as the results themselves, we introduce the following notations. For each connected subset I of the real axis we define $\mathcal{LP}(I)$ to be the class of all entire functions which can be uniformly approximated on disks about the origin by a sequence of real polynomials all of whose zeros lie in the set I . We shall use the notation $\mathcal{LP}(I)^*$ to denote the class of all entire functions that arise as products of real polynomials and functions in $\mathcal{LP}(I)$. It is obvious that $\mathcal{LP}(I) \subset \mathcal{LP}(I)^*$ and, for a nonconstant function φ in $\mathcal{LP}(I)^*$, φ is in the class $\mathcal{LP}(I)$ if and only if φ has no zeros outside I . From Rolle's theorem, the classes $\mathcal{LP}(I)^*$ and $\mathcal{LP}(I)$ are closed under differentiation and differentiation does not increase the number of nonreal zeros of a function in $\mathcal{LP}(\mathbf{R})^*$.

The purpose of this paper is to prove the following conjecture of Pólya [P2] and Wiman [Wil].

The Pólya-Wiman Conjecture. *For each $\varphi \in \mathcal{LP}(\mathbf{R})^*$, there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}(\mathbf{R})$.*

In fact, we will prove the following more general theorem.

Theorem 2. *Let I be a connected subset of the real axis. Then for each $\varphi \in \mathcal{LP}(I)^*$, there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}(I)$.*

Received by the editors May 17, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 30D15, 30D35.

Key words and phrases. Pólya-Wiman conjecture, Laguerre-Pólya class.

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 0002-9939/90 \$1.00 + \$.25 per page

Among the classes $\mathcal{LP}(I)$, $\mathcal{LP}(\mathbf{R})$ is called the Laguerre–Pólya class, since a classical theorem of Laguerre [L] and Pólya [P1] asserts that $\psi \in \mathcal{LP}(\mathbf{R})$ if and only if it can be expressed in the form

$$(1.1) \quad \psi(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_j \left(1 - \frac{z}{a_j}\right) e^{z/a_j},$$

where c, β, a_j are real, $\alpha \geq 0$, n is a nonnegative integer and $\sum_j a_j^{-2} < \infty$. (For a modern proof of this theorem, see Levin [Le, Chapter 8].) In the rest of this paper we will use the simpler notations $\mathcal{LP}$ and $\mathcal{LP}^*$ to denote the classes $\mathcal{LP}(\mathbf{R})$ and $\mathcal{LP}(\mathbf{R})^*$, respectively. From (1.1) we see that for $\varphi \in \mathcal{LP}^*$ the order of φ is at most 2 and the genus of φ is equal to 0, 1, or 2 (for definitions, see [B]).

In 1930 Pólya [P2] proved this conjecture for functions of order less than $\frac{2}{3}$, and later he extended his results to functions of order less than $\frac{4}{3}$ [P3]. On the other hand, in [Wi1] and [Wi2], Wiman established the validity of this conjecture for functions of the form $\exp(-\alpha z^2 + \beta z)\varphi(z)$, where $\alpha \geq 0$, $\beta \in \mathbf{R}$, and φ is of genus zero. Recently, Craven, Csordas, and Smith proved the conjecture for functions of genus at most 1 [CCS1, CCS2]. Therefore the only remaining case of the conjecture is for functions of genus 2.

In §2 we will prove the Pólya-Wiman conjecture (Theorem 1). Theorem 2 is a consequence of Theorem 1. In the proof of Theorem 1 we will require two technical results, Proposition 1 and Proposition 2, whose proofs will be given in §3.

2. PROOF OF THE THEOREMS

Let $\varphi \in \mathcal{LP}^*$. The following proposition states that if the Pólya-Wiman conjecture is true for φ , then the same is true for some functions in $\mathcal{LP}^*$ that are sufficiently close to φ .

Proposition 1. *Let $\{\psi_n\}$ be a sequence of functions in $\mathcal{LP}$ and assume that $\psi_n \rightarrow 1$ uniformly on compact sets in the plane. If $\varphi \in \mathcal{LP}^*$ and $\varphi^{(m)} \in \mathcal{LP}$ for some positive integer m , then we can find positive integers M and N such that*

$$(\varphi \psi_n)^{(M)} \in \mathcal{LP}, \quad n \geq N.$$

To complete the proof of Theorem 1, we need the following proposition.

Proposition 2. *Let $\varphi \in \mathcal{LP}^*$. If there is a positive integer n and real numbers γ_1 and γ_2 , $\gamma_1 < \gamma_2$, such that*

$$(D + \gamma_j)^n \varphi \in \mathcal{LP}, \quad j = 1, 2, \quad D = d/dz,$$

then there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}$.

Remark 2.1. (a) In later discussions it will be convenient to denote differentiation by D .

(b) Since the classes $\mathcal{LP}^*$ and $\mathcal{LP}$ are closed under multiplication and differentiation, and $e^{\gamma z} \in \mathcal{LP}$ for each real number γ , we see that these classes are closed under the operation $D + \gamma$ for each real number γ .

(c) Proposition 2 is a generalization of Theorem 2 of [CCS1].

Theorem 1. *For each $\varphi \in \mathcal{LP}^*$, there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}$.*

Proof. Let $\varphi \in \mathcal{LP}^*$ be given. In [Wil] Wiman proved that if φ has only a finite number of zeros, then our assertion is true. So we can assume that φ has infinitely many (real) zeros. Write

$$\varphi(z) = p(z)e^{-\alpha z^2 + \beta z} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{z/a_j},$$

where $p(z)$ is a real polynomial, $\alpha \geq 0$, $\beta, a_j \in \mathbf{R}$ and $\sum_j a_j^{-2} < \infty$. The function $p(z) \exp(-\alpha z^2 + \beta z)$ is in the class $\mathcal{LP}^*$ and has only a finite number of zeros. By Wiman's result, there is a positive integer m_1 such that $D^{m_1} p(z) \exp(-\alpha z^2 + \beta z) \in \mathcal{LP}$.

If we set $\psi_n(z) = \prod_{j=n+1}^{\infty} (1 - z/a_j) \exp(z/a_j)$, $n = 1, 2, \dots$, then each ψ_n is a function in the class $\mathcal{LP}$ and $\psi_n \rightarrow 1$ uniformly on compact sets in the plane. By Proposition 1,

$$D^{m_2} p(z) e^{-\alpha z^2 + \beta z} \prod_{j=N+1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{z/a_j} \in \mathcal{LP},$$

for some positive integers m_2 and N .

In [CCS1, p. 407], Craven, Csordas, and Smith proved the following lemma.

Lemma. *If $\varphi \in \mathcal{LP}^*$ and if $D^m \varphi \in \mathcal{LP}$ for some nonnegative integer m , then for any $a \in \mathbf{R}$, $D^{m+1}(z+a)\varphi(z) \in \mathcal{LP}$.*

Using this lemma, we have

$$D^{m_2+N} p(z) e^{-\alpha z^2 + \beta z} \prod_{j=N+1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{z/a_j} \prod_{j=1}^N \left(1 - \frac{z}{a_j}\right) \in \mathcal{LP},$$

that is,

$$D^{m_2+N} e^{\gamma_0 z} \varphi(z) \in \mathcal{LP},$$

where $\gamma_0 = -\sum_{j=1}^N a_j$. From Proposition 1, we get an $\varepsilon > 0$ and a positive integer m_3 such that $D^{m_3} \exp(\gamma z) \varphi(z) \in \mathcal{LP}$ whenever $|\gamma - \gamma_0| < \varepsilon$. Therefore we can find distinct real numbers γ_1 and γ_2 such that $D^{m_3} \exp(\gamma_j z) \varphi(z) \in \mathcal{LP}$, $j = 1, 2$, that is,

$$(D + \gamma_j)^{m_3} \varphi \in \mathcal{LP}, \quad j = 1, 2.$$

Finally, Proposition 2 gives the desired result. $\square$

Theorem 2. Let I be a connected subset of the real axis. Then for each $\varphi \in \mathcal{LP}(I)^*$, there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}(I)$.

Proof. First consider the case when $I = (-\infty, a]$, $a \in \mathbf{R}$. Let $\varphi \in \mathcal{LP}(-\infty, a]^*$. Since $\mathcal{LP}(-\infty, a]^* \subset \mathcal{LP}^*$, Theorem 1 implies that $\varphi^{(m_1)} \in \mathcal{LP}$ for some positive integer m_1 . From the fact that $\mathcal{LP}(-\infty, a]^*$ is closed under differentiation, we have $\varphi^{(m_1)} \in \mathcal{LP} \cap \mathcal{LP}(-\infty, a]^*$. Hence we can find a real number b , $a \leq b$, such that $\varphi^{(m_1)} \in \mathcal{LP}(-\infty, b]$.

It is well known [Le, Chapter 8] that for a nonconstant entire function f , $f \in \mathcal{LP}(-\infty, b]$ if and only if f can be represented in the form

$$f(z) = e^{\alpha z} g(z),$$

where $\alpha \geq 0$ and g is an entire function of genus zero which has no zeros outside the interval $(-\infty, b]$. From this result we see that if $\varphi^{(m_1)}$ has no zeros or is a polynomial, then our assertion is obvious.

Let $\varphi^{(m_1)} = \psi$ be transcendental, and assume that ψ has a zero. Then it is easy to see that each $\psi^{(n)}$, $n = 1, 2, \dots$, has at least one zero. Since each $\psi^{(n)}$ is in the class $\mathcal{LP}(-\infty, b]$, it must have the largest zero c_n , $n = 0, 1, 2, \dots$. Moreover we have $c_{n+1} \leq c_n$, $n = 0, 1, 2, \dots$.

We next claim that

$$(2.1) \quad \lim_{n \rightarrow \infty} c_n = -\infty.$$

Assume, on the contrary, that $\lim c_n = c > -\infty$. Let x be an arbitrary real number greater than c_0 . Since $\psi^{(n)}(c_n) = 0$, $n = 0, 1, 2, \dots$, we have

$$\psi(x) = \int_{c_0}^x \int_{c_1}^{t_1} \cdots \int_{c_{n-1}}^{t_{n-1}} \psi^{(n)}(t_n) dt_n \cdots dt_2 dt_1, \quad n = 2, 3, \dots,$$

and from this formula we get the following estimate

$$(2.2) \quad |\psi(x)| \leq \frac{M_n}{n!} (x - c)^n, \quad n = 1, 2, \dots,$$

where $M_n = \max\{|\psi^{(n)}(t)| : c \leq t \leq x\}$, $n = 1, 2, \dots$. On the other hand, Cauchy's integral formula gives positive numbers M and R which satisfy

$$(x - c) < R \quad \text{and} \quad \frac{M_n}{n!} \leq \frac{M}{R^n}, \quad n = 1, 2, \dots$$

Hence the right-hand side of (2.2) tends to zero as $n \rightarrow \infty$. Since x , $x > c_0$, is arbitrary we see that $\psi(x) = 0$ for $x > c_0$ and hence $\psi \equiv 0$. This contradiction shows that (2.1) is true.

Consequently we have proved that if $\varphi \in \mathcal{LP}(-\infty, a]^*$, then for any c , $c \leq a$, there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}(-\infty, c]$. Using the same argument, we can prove that if $\varphi \in \mathcal{LP}[a, \infty)^*$, then for any c , $c \geq a$, there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}[c, \infty)$. From these results our assertion is obvious. $\square$

Remark 2.2. The proof of Theorem 2 shows that for each $\varphi \in \mathcal{LP}(-\infty, a]^*$, the final set of φ (for the definition, see [P4]) is empty.

3. PROOF OF THE PROPOSITIONS

We begin this section with a brief review of those properties of functions in the class $\mathcal{LP}^*$ that will be used hereafter.

Let $\varphi \in \mathcal{LP}^*$, and let $\alpha_j \pm i\beta_j$, $j = 1, \dots, d$, be the nonreal zeros of φ . Then φ must be of the form

$$\varphi(z) = cz^n p(z) e^{-\alpha z^2 + \beta z} \prod_k \left(1 - \frac{z}{a_k}\right) e^{z/a_k},$$

where c , β , and a_k are real, $\alpha \geq 0$, n is a nonnegative integer, $\sum a_k^{-2} < \infty$, and $p(z) = \prod_{j=1}^d ((z - \alpha_j)^2 + \beta_j^2)$. A simple calculation shows that the imaginary part of the logarithmic derivative φ'/φ of φ cannot be equal to zero in the set

$$\{x + iy : y \neq 0 \text{ and } (x - \alpha_j)^2 + y^2 > \beta_j^2, j = 1, \dots, d\}.$$

Since $\mathcal{LP}^*$ is closed under differentiation, an inductive argument gives the following theorem.

Jensen-Nagy-Walsh Theorem. *Let $\varphi \in \mathcal{LP}^*$ and let $\alpha_j \pm i\beta_j$, $j = 1, \dots, d$, be the nonreal zeros of φ . If $z = x + iy$ is a nonreal zero of $\varphi^{(k)}$, then there is an integer j , $1 \leq j \leq d$, such that*

$$(x - \alpha_j)^2/k + y^2 \leq \beta_j^2.$$

Lemma 1. *Let $\psi \in \mathcal{LP}$ and assume that ψ is not a polynomial. Then for any positive number A there is a positive integer N such that $\psi^{(n)}$ has only simple zeros in the interval $(-A\sqrt{n}, A\sqrt{n})$ whenever $n \geq N$.*

Proof. For $\psi(z) = ce^{az^2}$ our assertion is trivial. If ψ is not of this form, we have $D(\psi'/\psi)(x) < 0$, whenever $x \in \mathbf{R}$ and $\psi(x) \neq 0$. Therefore, in any connected component of the set $\{x : x \in \mathbf{R}, \psi^{(k)}(x) \neq 0\}$, $\psi^{(k+1)}$ has at most one zero, counting multiplicities.

Now, to get a contradiction, assume that our assertion is not true. Then from the above argument, there is a positive number A and a strictly increasing sequence $\{n_j\}$ of positive integers such that in each interval $(-A\sqrt{n_j}, A\sqrt{n_j})$, there is a zero b_j of ψ whose multiplicity is greater than $n_j + 1$. Moreover we can assume that $b_i \neq b_j$ for $i \neq j$.

Let $a_1, a_2, \dots$ be the zeros of ψ . Since $\psi \in \mathcal{LP}$, we have

$$\infty > \sum_{a_j \neq 0} a_j^{-2} > \sum_{b_j \neq 0} (n_j + 1) b_j^{-2} > \sum (n_j + 1) (A\sqrt{n_j})^{-2} = \infty.$$

This is the desired contradiction. $\square$

Proposition 1. *Let $\{\psi_n\}$ be a sequence of functions in $\mathcal{LP}$ and assume that $\psi_n \rightarrow 1$ uniformly on compact sets in the plane. If $\varphi \in \mathcal{LP}^*$ and $\varphi^{(m)} \in \mathcal{LP}$ for some positive integer m , then we can find positive integers M and N such that*

$$(\varphi \psi_n)^{(M)} \in \mathcal{LP}, \quad n \geq N.$$

Proof. We may assume, without loss of generality, that all the nonreal zeros of φ , and hence those of $\varphi\psi_n$, $n = 1, 2, \dots$, are in the unit disk $|z| < 1$.

Since $\varphi^{(m)} \in \mathcal{LP}$, Lemma 1 gives a positive integer M , $M \geq m$, such that $\varphi^{(M)}$ has only simple zeros in the interval $(-\sqrt{M+1}, \sqrt{M+1})$. Let l be the number of (simple) zeros of $\varphi^{(M)}$ in $(-\sqrt{M+1}, \sqrt{M+1})$. From the intermediate value theorem, $(\varphi\psi_n)^{(M)}$ will have at least l zeros in $(-\sqrt{M+1}, \sqrt{M+1})$ as soon as n becomes sufficiently large.

On the other hand, the Jensen–Nagy–Walsh theorem states that the nonreal zeros of each $(\varphi\psi_n)^{(M)}$, $n = 1, 2, \dots$, must lie in the disk $|z| < \sqrt{M+1}$, and from Hurwitz's theorem, $\varphi^{(M)}$ and $(\varphi\psi_n)^{(M)}$ have the same number of zeros in $|z| < \sqrt{M+1}$ for all sufficiently large n .

Therefore, for all sufficiently large n , $(\varphi\psi_n)^{(M)}$ has exactly l zeros in $|z| < \sqrt{M+1}$, at least l real zeros in $|z| < \sqrt{M+1}$, and no nonreal zeros outside $|z| < \sqrt{M+1}$, that is, $(\varphi\psi_n)^{(M)} \in \mathcal{LP}$. $\square$

In the proof of Proposition 2, we will require the following lemmas.

Lemma 2. Let $\varphi \in \mathcal{LP}^*$ and assume that each $\varphi^{(n)}$, $n = 1, 2, \dots$, has exactly $2d$, $d > 0$, nonreal zeros. If there are real numbers γ_1 and γ_2 , $\gamma_1 < \gamma_2$, such that $(D + \gamma_j)\varphi \in \mathcal{LP}$, $j = 1, 2$, then there is a constant c such that each $\varphi^{(n)}$, $n = 1, 2, \dots$, has no zeros outside the strip $|\Im z| < c/n$.

This lemma, but with a restriction on the order of φ , has been proved by Craven, Csordas, and Smith [CCS1, (3.49), p. 428]. This lemma is proved by using the same argument as in the original proof, except that a refinement of [CCS1, Lemma 3, p. 411] is now required. The upper bound in that lemma must be recognized to be $2d+1$ when φ has infinitely many zeros. See [CCS2].

Lemma 3. Let f be an entire function of finite order and assume that $f^{(n)}(z_n) = 0$, $n = 0, 1, 2, \dots$. If there is a constant c such that

$$(3.1) \quad |z_{n-1} - z_n| < \frac{c}{n}, \quad n = 1, 2, \dots,$$

then $f \equiv 0$.

Proof. For an entire function g and a positive real number r , define $M(r; g)$, as usual, $M(r; g) = \max\{|g(z)| : |z| = r\}$. Since f is of finite order, there is a positive number A such that

$$(3.2) \quad M(r; f) \leq \exp(r^A),$$

for all sufficiently large r .

Fix a complex number z and let

$$t_n = |z| + |z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|, \quad n = 2, 3, \dots$$

Since $f^{(n)}(z_n) = 0$, $n = 0, 1, 2, \dots$, we get

$$f(z) = \int_{z_0}^z \int_{z_1}^{\zeta_1} \cdots \int_{z_{n-1}}^{\zeta_{n-1}} f^{(n)}(\zeta_n) d\zeta_n \cdots d\zeta_2 d\zeta_1, \quad n = 2, 3, \dots,$$

and from this formula we get

$$(3.3) \quad |f(z)| \leq \frac{M(t_n; f^{(n)})}{n!} t_n^n, \quad n = 2, 3, \dots$$

Let $\{R_n\}$ be an arbitrary sequence of real numbers such that $t_n < R_n$, $n = 1, 2, \dots$. From Cauchy's integral formula, we have

$$\frac{M(t_n; f^{(n)})}{n!} \leq \frac{M(R_n; f)}{(R_n - t_n)^n}, \quad n = 2, 3, \dots,$$

and from (3.2) and (3.3) we obtain

$$(3.4) \quad |f(z)| \leq \frac{\exp(R_n^4)}{(R_n/t_n - 1)^n}, \quad n = 2, 3, \dots,$$

whenever R_n becomes sufficiently large.

On the other hand, (3.1) implies that there is a constant B such that $t_n \leq B \log n$ for large n . If we set $R_n = (B \log n)^2$, $n = 2, 3, \dots$, then $t_n < R_n$ for large n , and a simple calculation shows that the right-hand side of (3.4) tends to zero as $n \rightarrow \infty$. Since z is arbitrary, this gives the desired result. $\square$

Remark 3.1. Lemma 3 is a special case of Theorem III of [G].

Proposition 2. Let $\varphi \in \mathcal{LP}^*$. If there is a positive integer n and two real numbers γ_1 and γ_2 , $\gamma_1 < \gamma_2$, such that

$$(D + \gamma_j)^n \varphi \in \mathcal{LP}, \quad j = 1, 2,$$

then there is a positive integer m such that $\varphi^{(m)} \in \mathcal{LP}$.

Proof. The following proof is based on an induction on n .

(I) Assume $\gamma_1 < \gamma_2$ and $(D + \gamma_j)\varphi \in \mathcal{LP}$, $j = 1, 2$. Suppose, on the contrary, that there is no integer m such that $\varphi^{(m)} \in \mathcal{LP}$. Since differentiation does not increase the number of nonreal zeros of a function in the class $\mathcal{LP}^*$, we may assume, without loss of generality, that each $\varphi^{(n)}$, $n = 0, 1, 2, \dots$, has exactly $2d$, $d > 0$, nonreal zeros. Then using the Jensen–Nagy–Walsh theorem, we obtain a sequence $\{z_n\}$ of complex numbers such that each z_n is a nonreal zero of $\varphi^{(n)}$ and

$$(3.5) \quad |z_{n+1} - z_n| \leq |\Im z_n|, \quad n = 0, 1, 2, \dots$$

From Lemma 2, there is a constant c such that $|\Im z_n| < c/n$, $n = 1, 2, \dots$. Hence, (3.5) implies that $|z_{n-1} - z_n| < 3c/n$, $n = 2, 3, \dots$. Since the order of φ is at most 2, Lemma 3 implies that $\varphi \equiv 0$. This is the desired contradiction.

(II) Suppose that our assertion is true for $1 \leq n \leq k-1$ and that $(D + \gamma_j)^k \varphi \in \mathcal{LP}$, $j = 1, 2$, for some real numbers γ_1 and γ_2 , $\gamma_1 < \gamma_2$. Since

$$(D + \gamma_j)^{k-1} (D + \gamma_1)(D + \gamma_2)\varphi \in \mathcal{LP}, \quad j = 1, 2,$$

the induction hypothesis implies that there is a positive integer m_1 such that

$$(D + \gamma_1)(D + \gamma_2)\varphi^{(m_1)} = D^{m_1}(D + \gamma_1)(D + \gamma_2)\varphi \in \mathcal{LP}.$$

Then we have

$$(D + \gamma_j)^{k-1} (D + \gamma_1) \varphi^{(m_1)} \in \mathcal{LP}, \quad j = 1, 2,$$

and

$$(D + \gamma_j)^{k-1} (D + \gamma_2) \varphi^{(m_1)} \in \mathcal{LP}, \quad j = 1, 2.$$

Again by the induction hypothesis, we get positive integers m_2 and m_3 such that

$$(D + \gamma_1) \varphi^{(m_1+m_2)}, (D + \gamma_2) \varphi^{(m_1+m_3)} \in \mathcal{LP}.$$

Let $m_4 = \max(m_1 + m_2, m_1 + m_3)$. Then

$$(D + \gamma_j) \varphi^{(m_4)} \in \mathcal{LP}, \quad j = 1, 2,$$

and hence there is a positive integer m , $m \geq m_4$, such that

$$\varphi^{(m)} \in \mathcal{LP}. \quad \square$$

ACKNOWLEDGMENT

I would like to thank Professors T. Craven, G. Csordas, and W. Smith and the referee for their helpful comments.

REFERENCES

- [B] R. P. Boas, *Entire functions*, Academic Press, New York, 1954.
- [CCS1] T. Craven, G. Csordas, and W. Smith, *The zeros of derivatives of entire functions and the Pólya-Wiman Conjecture*, Ann. of Math. **125** (1987), 405–431.
- [CCS2] ———, *Zeros of derivatives of entire functions*, Proc. Amer. Math. Soc. **101** (1987), 323–326.
- [G] W. Gontcharoff, *Recherches sur les dérivées successives des fonctions analytiques*, Ann. École Norm. **47** (1930), 1–78.
- [L] E. Laguerre, *Oeuvres I*, Gauthier-Villars, Paris, 1898.
- [Le] B. Ja. Levin, *Distribution of zeros of entire functions*, Trans. Math. Monographs, vol. 5, Amer. Math. Soc., Providence, RI, 1964.
- [P1] G. Pólya, *Über Annäherung durch Polynome mit lauter reellen Wurzeln*, Rend. Circ. Mat. Palermo **36** (1913), 279–295.
- [P2] ———, *Some problems connected with Fourier's work on transcendental equations*, Quart. J. Math. Oxford Ser. 1 (1930), 21–34.
- [P3] ———, *Über die Realität der Nullstellen fast aller Ableitungen gewisser ganzer Funktionen*, Math. Ann. **114** (1937), 622–634.
- [P4] ———, *On the zeros of derivatives of a function and its analytic character*, Bull. Amer. Math. Soc. **49** (1943), 178–191.
- [Wi1] A. Wiman, *Über eine asymptotische Eigenschaft der Ableitungen der ganzen Funktionen von den Geschlechtern 1 und 2 mit einer endlichen Anzahl von Nullstellen*, Math. Ann. **104** (1930), 169–181.
- [Wi2] ———, *Über die Realität der Nullstellen fast aller Ableitungen gewisser ganzer Funktionen*, Math. Ann. **114** (1937), 617–621.

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