ON THE NUMBER OF SUBGROUPS OF GIVEN ORDER IN A FINITE p -GROUP OF EXPONENT p

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ABSTRACT. A. Kulakoff [1] showed that a noncyclic p -group of order p^m , $p > 2$, contains $1 + p + kp^2$ subgroups of order p^n , $0 < n < m$, where k is a nonnegative integer. In this note we show that for $1 < n < m - 1$ a p -group of order p^m and exponent p contains $1 + p + 2p^2 + kp^3$ subgroups of order p^n .

1.

In this section G denotes a group of order p^m and exponent p . Let m, n be natural numbers. Let $\phi(G)$ denote the Frattini subgroup of G and $p^{d(G)} = p^d = |G : \phi(G)|$. In our situation $\phi(G) = G' = [G, G] = K_2(G)$. Let $K_3(G) = [K_2(G), G]$. For $i \in [1, d]$ let $M_i = \{H < G | \phi(G) \leq H, |G : H| = p^i\}$. Obviously $M_d = \{\phi(G)\}$. If $\mathfrak{M}$ is a certain set of subgroups of G and $F \leq G$, then $n(F)$ denotes the number of elements of $\mathfrak{M}$ which are subgroups of F . Obviously $n(G) = |\mathfrak{M}|$. We use the following enumeration principle of Philip Hall:

$$(1) \quad n(G) = \sum_{i=1}^d (-1)^{i-1} p^{i(i-1)/2} \sum_{H \in M_i} n(H),$$

This identity holds for arbitrary finite p -groups.

In this note we prove the following.

Theorem. *Let G be a p -group of order p^m and exponent p , $2 \leq n \leq m - 2$. Then the number of subgroups of order p^n in G is congruent to $1 + p + 2p^2$ modulo p^3 .*

We use the following

Lemma 1. *Let G be a p -group of order p^m and exponent p , $d = d(G)$.*

- (a) *If G is minimal non-Abelian then $m = 3$.*
- (b) *If $d = 2$ then $|G : K_3(G)| = p^3$.*

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- (c) Let H be a two-generator maximal subgroup of G . Then the number of two-generator maximal subgroups in G and $G/K_3(H)$ is the same.
- (d) If $\phi(G)$ is non-Abelian then it is not generated by two elements.
- (e) If $d > 2$ and G contains a two-generator maximal subgroup H then it contains exactly p^2 such subgroups.
- (f) If $d = 2$ and G contains a two-generator maximal subgroup then G contains exactly p such subgroups.

Proof. (a) Since G is non-Abelian it contains two noncommuting elements x, y and $G = \langle x, y \rangle$. Since $\phi(G) = G' = \langle [x, y] \rangle$ and $d = 2$ then $m = 3$.

(b) Obviously $G/K_3(G)$ is minimal non-Abelian; so $|G/K_3(G)| = p^3$ by (a).

(c) Let $D = K_3(H)$. Then $|H/D| = p^3$ by (b). Let F/D be a non-Abelian subgroup of index p in G/D . Suppose that $d(F) > 2$. Then $|F : \phi(F)| \geq p^3$. Obviously $F \neq H$. So $|G : F \cap H| = p^2$. Between $H \cap F$ and $\phi(F)$ there is a G -admissible subgroup F_1 of index p^2 in $H \cap F$. Since $|H/F_1| = p^3$ then $F_1 = D$. Since F/F_1 is Abelian and F/D is non-Abelian we obtain a contradiction. Hence $d(F) = 2$. Since $D < \phi(H) \leq \phi(G)$ our assertion is proved.

(d) Suppose that $d(\phi(G)) = 2$. Without loss we suppose that $K_3(\phi(G)) = 1$. Then $|\phi(G)| = p^3$ by (b). Let L be a G -admissible subgroup of order p^2 in $\phi(G)$. Then $|G : C_G(L)| = p$ and $\phi(G) < C_G(L)$, $\phi(G)$ is Abelian, a contradiction.

(e) In virtue of (c) we set $K_3(H) = 1$. Then $|G| = p^4$ by (b) so G is not extra-special. Hence $|Z(G)| = p^2$. Let $x \in Z(G) - \phi(G)$. Then $G = \langle x \rangle \times F$ with $F < G$ and our assertion is obvious.

(f) As in (e) we assume that $K_3(H) = 1$ where H is a two-generator maximal subgroup of G . Then $m = 4$ and $|M_1| = 1 + p$. Since $|Z(G)| = p$ then M_1 contains only one Abelian subgroup of index p , and (f) is proved.

Corollary 1. Let G be a p -group of order p^m and exponent p , $3 \leq n < m$. Then G contains kp two-generator subgroups of order p^n . Here k is nonnegative integer.

Proof. From Lemma 1(e,f) this assertion is true for $n = m - 1$. If $n < m - 1$ the result follows by induction using (1).

Remark. Let G be as in Corollary 1, $3 \leq n \leq m - 2$. If every two-generator subgroup of order p^s , $n < s \leq m$, contains a two-generator subgroup of index p then the number of two-generator subgroups of order p^n in G is divisible by p^2 . For the proof it is necessary to use Lemma 1(d).

Proof of the theorem. Suppose that the theorem is proved for all proper subgroups of G . Let $\mathfrak{M}$ be the set of all subgroups of order p^n in G . In virtue of (1) we have

$$(2) \quad n(G) \equiv \sum_{H \in \mathfrak{M}_1} n(H) - p \sum_{H \in \mathfrak{M}_2} n(H) \pmod{p^3}$$

(a) $n = m - 2$.

(1a) $d = 2$.

If G contains a two-generator maximal subgroup then it contains exactly p such subgroups (Lemma 1(f)). Hence

$$n(G) \equiv p(1+p) + 1(1+p+p^2) - p \equiv 1+p+2p^2 \pmod{p^3}.$$

(2a) $d > 2$.

If G contains a two-generator maximal subgroup then by (2) and Lemma 1(e) we have

$$n(G) \equiv p^2(1+p) + (1+p)(1+p+p^2) - p(1+p) \equiv 1+p+2p^2 \pmod{p^3}.$$

If G does not contain a two-generator maximal subgroup then

$$n(G) \equiv (1+p+p^2)(1+p+p^2) - p(1+p) \equiv 1+p+2p^2 \pmod{p^3}.$$

(b) $n < m - 2$. If $d = 2$, then by (2), Kulakoff's theorem and induction we have

$$n(G) \equiv (1+p)(1+p+2p^2) - p(1+p) \equiv 1+p+2p^2 \pmod{p^3}.$$

If $d > 2$ then $n(G) \equiv (1+p+p^2)(1+p+2p^2) - p(1+p)(1+p) \equiv 1+p+2p^2 \pmod{p^3}$. The theorem is proved.

Probably if $2 < n < m - 2$ and $d > 4$ then we have

$$n(G) \equiv 1+p+2p^2+2p^3 \pmod{p^4}$$

in the theorem.

2.

We apply some of the results of the previous section to irregular p -groups.

Lemma 2. Let H be a subgroup of order p^n and exponent p in a p -group G , $n < p$. If $\Omega_1(G) = \langle x \in G | x^p = 1 \rangle > H$ then $H < F \leq G$ where $|F| = p^{n+1}$ and $\exp F = p$.

Proof. Let $N = N_G(H)$. If x is an element of order p in $N - H$ then $\langle x, H \rangle$ has order $p^{n+1} \leq p^p$ so it is regular and $\exp \langle x, H \rangle = p$. In this case we set $F = \langle x, H \rangle$. If $H = \Omega_1(N)$ then H is characteristic in N so $N = G$, a contradiction, since $\Omega_1(N) = H < \Omega_1(G) = \Omega_1(N)$.

Corollary 2. Let G be an irregular p -group, $2 < n < p - \varepsilon$ where

$$\varepsilon = \begin{cases} 1, & G \text{ is of maximal class,} \\ 0, & \text{in the opposite case.} \end{cases}$$

Then the number of two-generator subgroups of order p^n and exponent p in G is divisible by p .

Proof. Let $\mathfrak{M} = \{B_1, \dots, B_l\}$ be the set of all two-generator subgroups of order p^n and exponent p in G . Let $\mathfrak{N} = \{A_1, \dots, A_r\}$ be the set of all subgroups of

order p^{n+1} and exponent p in G such that every A_i contains a two-generator subgroup of index p . In virtue of Lemma 2 if the set $\mathfrak{M}$ is nonempty then $\mathfrak{N}$ is nonempty also. Let a_i be the number of elements of $\mathfrak{M}$ contained in A_i , $i \in [1, r]$, and b_j be the number of elements of $\mathfrak{N}$ which contain B_j . Then

$$(3) \quad a_1 + \cdots + a_r = b_1 + \cdots + b_t.$$

By Corollary 1 we have $a_i \equiv 0 \pmod{p}$ for all $i \in [1, r]$. Let $N = N_G(B_j)$. Obviously all elements of $\mathfrak{N}$ containing B_j are subgroups of N . We prove that $b_j \equiv 1 \pmod{p}$. We may assume without loss that $N = G$. If X is an element of order p in $G - B_j$ then $\langle x, B_j \rangle$ has order p^{n+1} and exponent p , and, obviously, $\langle x, B_j \rangle$ contains

$$(1 + p + \cdots + p^n) - (1 + p + \cdots + p^{n-1}) = p^n$$

subgroups of order p which lie outside B_j . By [2] G contains $1 + p + \cdots + p^n + kp^{n+1}$ subgroups of order p , where $k \geq 0$ is an integer. Hence G contains $p^n + kp^{n+1}$ subgroups of order p which are not subgroups of B_j . So

$$b_j = (p^n + kp^{n+1})/p^n = 1 + kp \equiv 1 \pmod{p}.$$

Hence $t \equiv b_1 + \cdots + b_t = a_1 + \cdots + a_r \equiv 0 \pmod{p}$, and our assertion is proved.

Lemma 1(a) has the following.

Corollary 3. *Let G be a finite p -group, $\mathfrak{M} = \{H \leq G \mid |H| \leq p^4, \exp H \leq p^2\}$. If all elements of $\mathfrak{M}$ are Abelian then $\Omega_1(G)$ is Abelian and $C_G(\Omega_1(G)) \geq \Omega_2(G)$.*

Proof. We note that all subgroups of exponent p in G are Abelian by Lemma 1(a).

Let H be a normal subgroup of exponent p of maximal order in G . Suppose that $\Omega_1(G) - H$ contains an element x of order p . Then $L = \langle x, H \rangle$ is non-Abelian of exponent p^2 by choice of H . Let A be a minimal non-Abelian subgroup of L . Then $HA = L$ and $H \cap A$ is elementary of index p in A . Since $|A'| = p$ and A/A' is two-generator (see proof of Lemma 1(a)) then $|A \cap H| \leq p^3$, $|A| \leq p^4$, $\exp A \leq p^2 \Rightarrow A \in \mathfrak{M}$, a contradiction. Hence $H = \Omega_1(G)$ is elementary Abelian.

Let y be an element of order p^2 in G . Then $M = \langle y, H \rangle$ has exponent p^2 , $|M : H| = p$, since $y^p \in H$. As in the preceding paragraph we obtain a contradiction. Hence M is Abelian and $\Omega_2(G) \leq C_G(H)$, and the corollary is proved.

Lemma 3. *Let G be a finite p -group of order p^m , $m \geq p + 2$.*

(a) [3]. *If G contains a maximal subgroup H which is absolutely regular (i.e. $|H : \langle x^p \mid x \in H \rangle| < p^p$), and if G is not absolutely regular or of maximal class then $G = H\Omega_1(G)$.*

(b) [2, 4]. *Suppose $p + 1 < n < m$, G is not of maximal class. Then the number of subgroups of maximal class and order p^n in G divisible by p^2 .*

(c) If G is of maximal class then the number of subgroups of order p^{p-1} and exponent p in G is equal to $1 + kp^2$, where k is a nonnegative integer. In this case G does not contain a normal subgroup of order p^p and exponent p .

(d) If G does not contain a normal subgroup of order p^p and exponent p then G is absolutely regular or of maximal class.

Assertions (c) and (d) are proved in [3].

Corollary 4. Suppose that the irregular p -group G is not of maximal class, $|G| = p^m$, $m > p + 2$. If $\mathfrak{M} = \{H < G \mid |H| = p^{p-1}, \exp H = p\}$ then $|\mathfrak{M}| \equiv 1 + p \pmod{p^2}$.

Proof. By Lemma 3(d) the group G contains a normal subgroup L of order p^p and exponent p . If $L = \Omega_1(G)$ then our assertion follows from Kulakoff's theorem (which is true for noncyclic groups of exponent 2 also). In virtue of Lemma 3(a) G has no absolutely regular maximal subgroup. Suppose the corollary is true for all proper subgroups of G . By (1) we obtain

$$(4) \quad n(G) \equiv \sum_{H \in \mathfrak{M}_1} n(H) - p \pmod{p^2}.$$

If M_1 does not contain an element of maximal class then by induction and (4) we have

$$n(G) \equiv (1 + p)(1 + p) - p \equiv 1 + p \pmod{p^2}.$$

If M_1 contains an element of maximal class then it contains exactly p^2 such elements (Lemma 3(b)) and for any such subgroup F we have $n(F) \equiv 1 \pmod{p^2}$. Hence by (4) and induction we have

$$n(G) \equiv p^2 + (1 + p)(1 + p) - p \equiv 1 + p \pmod{p^2},$$

and the corollary is proved.

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