ON L^2 -PLURIGENERA OF NOT-LOG-CANONICAL GORENSTEIN ISOLATED SINGULARITIES

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ABSTRACT. We will give an application of S. Izumi's theorems of order functions of analytic spaces to classification problems of isolated singularities by means of plurigenus δ_m .

INTRODUCTION

In his works [6, 7, 8, 9], S. Izumi had shown several fundamental results on the analytic orders and the convergence of formal functions on analytic spaces. The purpose of this short note is an application of his theory to the study of L^2 -plurigenus of complex analytic isolated singularity. Let (V, p) be a normal d -dimensional isolated singularity over $\mathbb{C}$ and $\psi: (X, A) \rightarrow (V, p)$ a resolution of singularity (V, p) with the exceptional locus $A = \psi^{-1}(p)$. We say a resolution ψ is good if A is a divisor of normal crossings on X . Then the m th L^2 -plurigenus of (V, p) is the integer $\delta_m(V, p)$ which was introduced in [17] and can be computed as

$$\delta_m(V, p) = \dim(\omega_V^{[m]} / \psi_*(\omega_X^{[m]}((m-1)A_{\text{red}}))).$$

Our result of this note is the following.

Theorem. *Let (V, p) be a normal d -dimensional complex analytic isolated singularity whose canonical module ω_V is $\mathbb{Q}$ -Cartier of index r . If (V, p) is not a log-canonical singularity, then $\limsup_{n \rightarrow \infty} \delta_n/n^d$ is positive.*

This means that not-log-canonical Gorenstein isolated singularities are of general type in the sense of L^2 -plurigena. The finiteness of $\limsup_{n \rightarrow \infty} \delta_n/n^d$ had been shown in [17] and now is well known. Further, there are several interesting studies on L^2 -plurigena (e.g., Watanabe [18], S. Ishii [2, 3, 4, 13]). Among others we will give attention to the following Morales' assertion in connection with our present paper.

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Theorem (Morales [13]). *Let $R = \bigoplus_{k \geq 0} R_k$ be a normal graded d -dimensional Gorenstein ring with $R_0 = \mathbb{C}$ and define the isolated singularity $\text{Spec}(R) = V$ at $p = V(R_+)$. Then the m th L^2 -plurigenus is given as $\delta_m(R) = \sum_{k \geq 0}^{m \cdot a(R)} \dim R_k$.*

Here the integer $a(R)$ is defined as $a(R) = \max\{\alpha \in \mathbb{Z} \mid [H_{R_+}^d(R)]_\alpha \neq 0\}$ following Goto–Watanabe [1].

Let us recall the definition of log-canonical singularity. Let (V, p) be a singularity with $\mathbb{Q}$ -Cartier canonical module ω_V ; there is an integer $r \geq 1$ where $\omega_V^{[r]}$ is generated by a meromorphic r -ple d -form ω_0 . Let $\psi: (X, A) \rightarrow (V, p)$ be a resolution of singularity with $A = \psi^{-1}(p) = \bigcup_{j=1}^s A_j$ the exceptional divisor. Let us write the divisor as $(1/r) \text{div}_X(\psi^* \omega_0) = \sum_{j=1}^s a_j A_j$ with $a_j \in \mathbb{Q}$. Then (V, p) is a *log-canonical* singularity if $a_j \geq -1$ for any j . In the situation of the Theorem of Morales, (V, p) is log-canonical if and only if $a(R) \leq 0$. Hence our main theorem follows from Morales' theorem if (V, p) has a good $\mathbb{C}^*$ -action.

For further related results and general treatments about log-canonical singularity, we refer to [10, 16, 3].

As is seen in the above, the log-canonical condition is given by the conditions for all divisors A_j . Hence not-log canonical singularity has at least one component A_1 where the condition $a_1 \geq -1$ does not hold. We must control the behaviour of the order functions $\nu_{A_i}(\psi^*(f\omega_0))$ for all A_i and all $f \in \mathcal{O}_{V,p}$ from the data concerning A_1 . So there were some difficulties on such a problem. Fortunately we can overcome them by Izumi's theory.

1. REES' FORMULATION OF IZUMI'S THEOREM

We will prove the following.

(1.1) **Theorem** (S. Izumi). *Let $\psi: (X, A) \rightarrow (V, p)$ be a proper birational morphism of normal complex analytic spaces with $\psi^{-1}(p)_{\text{red}} = A$. Let $A = (\bigcup_{j=1}^m A_j) \cup A'$ be the decomposition such that A_j is an irreducible Weil divisor on X for $1 \leq j \leq m$ and the codimension of A' is not less than two in X .*

Then there is a positive constant c_ψ which satisfies the relation $\nu_{A_j}(\psi^(f)) \leq c_\psi \nu_{A_i}(\psi^*(f))$ for any f of $\mathcal{O}_{V,p}$ and for any i and j .*

The formulation of (1.1) is similar to the version due to Professor D. Rees [14, 15] who calls it “a convenient form of Izumi's Theorem” (see also [7, Theorem (1.2)] in the case $d = 2$). However in the case [14, 15], the morphism ψ should be projective and the base field may be more general. So we will present a proof of (1.1) as a corollary of the theorem which we shall call “ CI_2 ”. The arguments below were suggested to the authors by S. Izumi.

(1.2) **Theorem** (CI_2 : [7, Theorem (3.4); 6]). *For any morphism $\Phi: (Y, \eta) \rightarrow (X, \xi)$ with the geometric rank $\text{grnk}_\eta \Phi = \dim X_\xi$, there exist $a = a(\Phi) \geq 1$,*

$b = b(\Phi) \geq 0$ such that

$$\nu_{\eta}(\Phi^*(f)) \leq a\nu_{\xi}(f) + b,$$

where the geometric rank $\text{grnk}_{\eta}\Phi$ is the integer defined by; $\text{grnk}_{\eta}\Phi = \frac{1}{2} \inf\{\text{topological dimension of } \Phi(U); U \text{ in a neighborhood of } \eta\}$.

Proof of (1.1). (cf. Proof of Corollary (3.5) in [7], and Theorem I of [9]). Take a point $\eta_i \in A_i$ for each i . Let $\psi_i: (X, \eta_i) \rightarrow (V, p)$ be the restriction of ψ to the germ (X, η_i) for each i . Now ψ_i satisfies $\text{grnk}_{\eta_i}\psi_i = \dim X_{\eta_i}$. Hence there are constants $a_i \geq 1$, and $b_i \geq 0$ where $\nu_{\eta_i}(\psi^*(f)) \leq a_i\nu_p(f) + b_i$ holds for any $f \in O_{V,p}$. Let the reflexive hull of mO_X be denoted by $(mO_X)^{**} = O_X(-\sum_{j=1}^m y_j A_j)$ with $y_j \in \mathbf{Z}$. Here y_j is written as $y_j = \min_{f \in mO_{V,p}} \nu_{A_j}(\psi^*(f))$ for each j . We will show

$$(1.3) \quad \left(\frac{a_i}{y_j} + b_i \right) \nu_{A_j}(\psi^*(f)) \geq \nu_{A_i}(\psi^*(f))$$

holds for any i and j and for any $f \in O_{V,p}$.

We have the relations

$$a_i(\nu_{A_i}(\psi^*(f)) - b_i)/a_i + b_i = \nu_{A_i}(\psi^*(f)) \leq \nu_{\eta_i}(\psi^*(f)) \leq a_i\nu_p(f) + b_i$$

by Theorem (1.2). Hence $\nu_p(f) \geq (\nu_{A_i}(\psi^*(f)) - b_i)/a_i$. Set the integer p_i as

$$p_i = \min\{\alpha \in \mathbf{Z} | \alpha \geq (\nu_{A_i}(\psi^*(f)) - b_i)/a_i\}.$$

Then $\psi^*(f) \in m^{p_i}O_X \subset O_X(-p_i \sum_{j=1}^m y_j A_j)$. This means $\nu_{A_i}(\psi^*(f)) \geq y_j p_i \geq y_j(\nu_{A_j}(\psi^*(f)) - b_j)/a_j$, verifying (1.3).

If we set c_{ψ} as

$$c_{\psi} = \max_{i, j \text{ with } i \neq j} \left(\frac{a_i}{y_j} + b_i \right),$$

then we obtain (1.1) by (1.3). $\square$

(1.4) *Remark.* In the proof of (1.3), we can choose an element $f \in m$ such that $\nu_{A_j}(\psi^*(f)) = y_j$ holds for any j . Hence (1.3) implies the relation $a_i + b_i \geq y_i$ for every i .

2. PROOF OF MAIN THEOREM

In what follows, we assume the condition $\omega_{V,p} \cong O_{V,p}$ (cf. (2.1)). Let $\psi: (X, A) \rightarrow (V, p)$ be a good resolution of the singularity (V, p) such that mO_X is an invertible O_X -module sheaf. Let $A = \bigcup_{j=1}^s A_j$ be the decomposition of the exceptional locus $\psi^{-1}(p) = A$ into the irreducible components. Represent mO_X as $mO_X = O_X(-\sum_{j=1}^s y_j A_j)$ with $y_j \in \mathbf{Z}$ (cf. Proof of (1.1)).

If we choose a meromorphic d -form ω_0 as $O_V \omega_0 = \omega_V$, then we obtain the isomorphisms

$$\omega_V^{[m]} / \psi_*(\omega_X^{[m]}((m-1)A)) \cong (O_V / J^{(m)})(\omega_0)^{\otimes m},$$

where $J^{(m)} = \text{Ann}_{O_X}(\omega_V^{[m]} / \psi_*(\omega_X^{[m]}((m-1)A)))$ for $m \geq 1$. If we denote the divisor of ω_0 as $\text{div}_X(\omega_0) = \sum_{j=1}^s a_j A_j$, then we can easily see the relation $J^{(m)} = \psi_*(O_X(m \sum_{j=1}^s (a_j + 1)A_j - \sum_{j=1}^s A_j)) \subset O_V$ for $m \geq 1$.

Now we assume that (V, p) is not log-canonical. Hence there is an index i_0 of $\{1, \dots, s\}$ such that $a_{i_0} < -1$. For simplicity we assume $i_0 = 1$. Let $f \in J^{(m)}$. Then we have

$$-m(a_1 + 1) + 1 \leq \nu_{A_1}(\psi^*(f)) \leq c_\psi \nu_{A_j}(\psi^*(f))$$

for each $j \geq 2$, where c_ψ is the constant by Izumi's Theorem (1.1). Hence $J^{(m)} O_X \subset O_X((m(a_1 + 1) - 1)A_1 + \sum_{j \geq 2} (m(a_1 + 1) - 1)A_j / c_\psi)$.

If the conditions

$$(*)_{m,k} \quad -m(a_1 + 1) + 1 \geq k y_1 \quad \text{and} \quad -m(a_1 + 1) + 1 \geq k c_\psi y_j \quad \text{for } j \geq 2$$

hold, then we obtain the relation $J^{(m)} O_X \subset O_X(-k \sum_{j=1}^s y_j A_j) = \overline{m^k O_X}$. In this case we obtain $J^{(m)} \subset \psi_*(J^{(m)} O_X) \subset \psi_*(\overline{m^k O_X}) = \overline{m^k}$. Here $\overline{m^k}$ is the integral closure of m^k [12].

Now set the integer D as $D = \max\{y_1, c_\psi y_2, \dots, c_\psi y_s\}$. Then the conditions $(*)_{m,k}$ hold for a couple (m, k) with the relation $m = k\{D/(-a_1 - 1)\}$. Here, for a member α , $\{\alpha\}$ means $\min\{\beta \in \mathbf{Z} | \alpha \leq \beta\}$. Therefore we obtain $J^{(k\{D/(-a_1-1)\})} \subset \overline{m^k}$ and the inequality $\delta_{k\{D/(-a_1-1)\}} \geq \dim O_V(\overline{m^k})$. Hence

$$\limsup_{n \rightarrow \infty} \delta_n / n^d \geq \{D/(-a_1 - 1)\}^{-d} \limsup_{n \rightarrow \infty} \dim(O_V / (\overline{m^n})) / n^d.$$

It is well known that $\lim_{n \rightarrow \infty} \dim(O_V / (\overline{m^n})) / n^d$ equals $(d!)^{-1}$ (the multiplicity of (V, p)). $\square$

(2.1) *Remark.* If the canonical module of (V, p) is $\mathbf{Q}$ -Cartier of index r , the arguments above implies the inequality $\lim_{n \rightarrow \infty} \delta_{rn} / n^d > 0$.

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REFERENCES

1. S. Goto and Keiichi Watanabe, *On graded rings I*, J. Math. Soc. Japan. **30** (1978), 179–213.
2. S. Ishii, *On isolated Gorenstein singularities*, Math. Ann. **270** (1985), 541–554.
3. ———, *Isolated $\mathbf{Q}$ -Gorenstein singularities of dimension three*, Adv. Stud. Pure Math. **8** (1986), 165–198.

4. S. Ishii, *Small deformation of normal singularities*, Math. Ann. **275** (1986), 139–148.
5. ———, *The asymptotic behavior of the growth of plurigenera for a normal isolated singularity*, Math. Ann. (to appear).
6. S. Izumi, *Linear complementary inequalities for orders of germs of analytic functions*, Invent. Math. **65** (1982), 459–471.
7. ———, *A measure of integrity for local analytic algebras*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **21** (1985), 719–735.
8. ———, *Gabrielov's rank condition is equivalent to an inequality of reduced orders*, Math. Ann. **276** (1986), 81–89.
9. ———, *The rank condition and convergence of formal functions*, Duke Math. J. **59** (1989), 241–264.
10. Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. **127** (1988), 93–163.
11. D. T. Lê, *Algebraic methods in the study of singularities*, Lecture Notes Tokyo University Komaba, 1977.
12. M. Lejeune-Jalabert and B. Teisser, *Cloture intégrale des ideaux et equisingularite*, preprint, Ecole Polytechnique, 1974.
13. M. Morales, *Resolution of quasi-homogeneous singularities and plurigeneras*, Compositio Math. **64** (1987), 311–327.
14. D. Rees, *Izumi's theorem*, preprint, 1985.
15. ———, *Izumi's theorem*, Lectures given at Berkeley: Microprogram on Commutative Algebra, 1987; Publ. M.S.R.I. **15**, Springer-Verlag, New York and Berlin, 1989, pp. 407–416.
16. F. Sakai, *Kodaira dimensions of complements of divisors*, in Complex Analysis and Algebraic Geometry: A collection of papers dedicated to K. Kodaira, Iwanami, Tokyo, 1977, pp. 239–257.
17. Kimio Watanabe, *On plurigenera of normal isolated singularities I*, Math. Ann. **250** (1980), 65–94.
18. ———, *On plurigenera of normal isolated singularities II*, Adv. Stud. Pure Math. **8** (1986), 671–685.

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