WEAKLY FACTORIAL DOMAINS AND GROUPS OF DIVISIBILITY

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ABSTRACT. An integral domain R is said to be weakly factorial if every nonunit of R is a product of primary elements. We give several conditions equivalent to R being weakly factorial. For example, we show that the following conditions are equivalent: (1) R is weakly factorial; (2) every convex directed subgroup of the group of divisibility of R is a cardinal summand; (3) if P is a prime ideal of R minimal over a proper principal ideal (x) , then P has height one and $(x)_P \cap R$ is principal; (4) $R = \cap R_P$, where the intersection runs over the height-one primes of R , is locally finite, and the t -class group of R is trivial.

Throughout this note, R will be a commutative integral domain with identity having quotient field K . Suppose that R is a UFD. If S is a saturated multiplicatively closed subset of R , then $S = \{\lambda p_1 \cdots p_n \mid \lambda \text{ a unit of } R, n \geq 0, \text{ each } (p_i) \in Y\}$ for some subset $Y \subseteq X^{(1)}$, where $X^{(1)}$ is the set of height-one prime ideals of R . Conversely, every such subset $Y \subseteq X^{(1)}$ determines a saturated multiplicatively closed subset of R . If we set $T = \{\lambda p_1 \cdots p_n \mid \lambda \text{ a unit of } R, n \geq 0, \text{ each } (p_i) \in X^{(1)} - Y\}$, then every nonzero element r of R may be written uniquely up to units in the form $r = st$ where $s \in S$ and $t \in T$. Stated in terms of the group of divisibility $G(R)$ of R , $\langle S \rangle$ is a cardinal summand of $G(R)$. In fact, $\langle S \rangle \oplus_{\mathbb{C}} \langle T \rangle = G(R)$. (See the next paragraph for definitions.) Here $G(R)$ is a cardinal sum of copies of $\mathbb{Z}$, one for each height-one prime, and $\langle S \rangle$ is the sum of the cyclic summands corresponding to the primes from Y . The purpose of this paper is to characterize the integral domains with this property. We show that an integral domain R has the property that every convex directed subgroup of $G(R)$ is a cardinal summand of $G(R)$ if and only if every nonunit of R is a product of primary elements, that is, R is weakly factorial.

Let R be an integral domain with quotient field K . The *group of divisibility of R* is the group $G(R) = K^*/U(R)$, where K^* is the multiplicative group of K and $U(R)$ is the group units of R . $G(R)$ is partially ordered

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by $xU(R) \leq yU(R) \Leftrightarrow x|y$ in R . For results on partially ordered abelian groups and groups of divisibility, the reader is referred to [6, Chapter 3; 10]. If S is a saturated multiplicatively closed subset of R , then it is easily seen that $\langle S \rangle = \{s_1 s_2^{-1} U(R) | s_1, s_2 \in S\}$ is a convex directed subgroup of $G(R)$. Conversely, Mott [10, Theorem 2.1] has shown that every convex directed subgroup of $G(R)$ has this form. A subgroup H of $G(R)$ is called a *cardinal summand* of $G(R)$ if there is a convex directed subgroup L of $G(R)$ with $G(R) = H \oplus_{\mathbb{C}} L$, that is, $G(R)$ is the algebraic direct sum of H and L and the partial order on $G(R) = H \oplus L$ is given by $(h_1, l_1) \leq (h_2, l_2) \Leftrightarrow h_1 \leq h_2$ and $l_1 \leq l_2$. Mott and Schexnayder [11, Proposition 4.1] have shown that a saturated multiplicatively closed subset S of R has $\langle S \rangle$ as a cardinal summand of $G(R)$ if and only if there is a saturated multiplicatively closed subset T of R with $S \cap T = U(R)$ such that every nonzero element r of R can be written uniquely up to units in the form $r = st$, where $s \in S$, $t \in T$ and each element s of S is v -coprime to each element $t \in T$ in the sense that $(s, t)_v = R$ or, equivalently, that $(s) \cap (t) = (st)$. Here v denotes the v -operation, that is, for an ideal I of R , $I_v = (I^{-1})^{-1} = [R : [R : I]]$.

An element $x \in R$ is said to be *primary* if (x) is a primary ideal of R and an integral domain R is said to be *weakly factorial* if every (nonzero) nonunit of R is a finite product of primary elements. Weakly factorial domains were introduced in [2]. Examples of weakly factorial domains include UFD's, one-dimensional semi-quasi-local domains, and GCD domains which are generalized Krull domains. It was shown that, for R weakly factorial, $R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite. Also, a Krull domain is factorial if and only if it is weakly factorial. Hence, a weakly factorial Noetherian domain is factorial if and only if it is integrally closed.

For a Krull domain R , the intersection $R = \bigcap_{P \in X^{(1)}} R_P$ is locally finite, and R is factorial if and only if $Cl(R) = 0$ where $Cl(R)$ is the divisor class group of R . Since the divisorial ideals form a group under the v -product $(A*B = (AB)_v)$ if and only if R is completely integrally closed, $Cl(R)$ need not be a group for an arbitrary integral domain. However, for any integral domain R , there is a group $Cl_t(R)$ called the t -class group of R which agrees with $Cl(R)$ when R is a Krull domain. A nonzero fractional ideal I of R is said to be *t -invertible* if there is a fractional ideal B of R with $(AB)_t = R$. Here t denotes the t -operation given by $I_t = \bigcup \{(a_1, \dots, a_n)_v | 0 \neq (a_1, \dots, a_n) \subseteq I\}$. An ideal I is said to be a *t -ideal* if $I_t = I$. The t -operation is an example of a finite-character star operation. For results on star operations, the reader is referred to [6, §§32 and 34]. For any integral domain R , the set of t -invertible t -ideals forms a group under the t -product $A*B = (AB)_t$. The *t -class group* of R is $Cl_t(R)$ —the group of t -invertible t -ideals modulo the subgroup of principal fractional ideals. For results on the t -class group, the reader is referred to [4, 5]. We show that, for a domain R , R is weakly factorial if and only if $R = \bigcap_{P \in X^{(1)}} R_P$ is locally finite and $Cl_t(R) = 0$. This result is part of the next theorem, the main result of this paper.

Theorem. *Let R be an integral domain with quotient field K . Then the following statements are equivalent:*

- (1) *Every convex directed subgroup of $G(R)$ is a cardinal summand of $G(R)$.*
- (2) *For each saturated multiplicatively closed subset S of R , $\langle S \rangle$ is a cardinal summand of $G(R)$.*
- (3) *For each prime ideal P of R , $\langle R - P \rangle$ is a cardinal summand of $G(R)$.*
- (4) *R is weakly factorial, that is, every nonunit of R is a product of primary elements.*
- (5) *$R = \bigcap_{P \in X^{(1)}} R_P$ where the intersection is locally finite and the natural map $G(R) \rightarrow \bigoplus_{P \in X^{(1)}} G(R_P)$ is surjective (and hence an order isomorphism).*
- (6) *If P is a prime ideal of R minimal over a proper principal ideal (x) , then $htP = 1$ and $(x)_P \cap R$ is principal.*
- (7) *$R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite and $Cl_t(R) = 0$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from the previously mentioned result of Mott [10, Theorem 2.1] that every convex directed subgroup of $G(R)$ has the form $\langle S \rangle$ for some saturated multiplicatively closed subset S of R , and conversely. (2) $\Rightarrow$ (3). Clear. (3) $\Rightarrow$ (6). Let P be a prime ideal minimal over (x) . Then P is a t -ideal. Suppose that there is a prime ideal Q with $0 \neq Q \subsetneq P$. Let $S = R - Q$. Then $\langle S \rangle$ is a cardinal summand of $G(R)$. By [11, Proposition 4.1] there is a multiplicatively closed subset T of R with $S \cap T = U(R)$, $ST = R - \{0\}$, and, for $s \in S$, and $t \in T$, $(s, t)_v = R$. Now $P \cap S \neq \emptyset$, so there exists an $s_0 \in P \cap S$. Let $0 \neq q \in Q$, so $q = st$ where $s \in S$ and $t \in T$. Then $s \notin Q$, so $t \in Q$. Thus $s_0, t \in P$ and hence $R = (s_0, t)_v \subset P_t = P$, a contradiction. Therefore, $htP = 1$. (This shows that, in a weakly factorial domain, every prime t -ideal has height one.) It follows from [11, Proposition 4.1], taking $S = R - P$, that $(x)_P \cap R$ is principal. (6) $\Rightarrow$ (4). Let S be the multiplicatively closed subset of R consisting of all products of primary elements and units. We first show that S is saturated. Let $x \in S$ be a nonunit and y be a nonunit factor of x . If $x = q_1 \cdots q_r$, where (q_i) is P_i -primary, then $\{P_1, \dots, P_r\}$ is the set of height-one primes containing (x) . Hence the set of height-one primes containing (y) is a subset of $\{P_1, \dots, P_r\}$, say $\{P_1, \dots, P_n\}$. Now $(y)_{P_1} \cap R$ is principal, say $(y)_{P_1} \cap R = (q'_1)$. Since $\dim R_{P_1} = 1$, (q'_1) is P_1 -primary. Now $y \in (q'_1)$, so $y = q'_1 t_1$ where $t_1 \notin P_1$, but $t_1 \in P_2 \cap \cdots \cap P_n$. Continuing in this manner, we get that $y = q'_1 \cdots q'_n t_n$, where (q'_i) is P_i -primary and t_n is contained in no height-one prime ideal, hence a unit. Suppose that some nonunit $a \in R$ is not in S . Then $(a) \cap S = \emptyset$ since S is saturated. Hence (a) can be enlarged to a prime ideal $P \supseteq (a)$ maximal with respect to missing S . Shrinking P to a prime ideal Q minimal over (a) , we have a height-one prime ideal Q containing no primary elements. But this is absurd, for the ideal $(a)_Q \cap R$ is itself a principal primary ideal contained in Q . (4) $\Rightarrow$ (2). Let $\bar{R}$ be a weakly factorial integral domain and let S be a saturated multiplicatively closed subset of $\bar{R}$. Let $x \in S$ be a

nonunit and suppose that $x = q_1 \cdots q_n$ where (q_i) is P_i -primary. Let (q) be P_i -primary. Then, for some $s \geq 1$, $q_i^s \in (q)$, so $q_i^s = rq$ for some $r \in R$; hence q is a factor of $x^s \in S$. Since S is saturated, $q \in S$. Thus it is easily seen that $S = \{\lambda q_1 \cdots q_n \mid \lambda \in U(R), n \geq 0, \text{ each } (q_i) \text{ is } P_i\text{-primary where } P_i \in X^{(1)} \text{ with } P_i \cap S \neq \emptyset\}$. Let $T = \{\lambda q_1 \cdots q_n \mid \lambda \in U(R), n \geq 0, (q_i) \text{ is } P_i\text{-primary where } P_i \cap S = \emptyset\}$. Then T is a saturated multiplicatively closed subset of R satisfying the conditions of part (2) of [11, Proposition 4.1]. Hence by this proposition, $\langle S \rangle$ is a cardinal summand of $G(R)$. (4) $\Rightarrow$ (5). By [2, Corollary 14], $R = \bigcap_{P \in X^{(1)}} R_P$ and this intersection is locally finite. Let $0 \neq t \in R_P$ be a nonunit where $P \in X^{(1)}$. Then $t = qu$ where (q) is P -primary and u is a unit of R_P . Thus $qU(R) \rightarrow tU(R_P)$. It easily follows that the natural map $G(R) \rightarrow \bigoplus_{P \in X^{(1)}} G(R_P)$ is surjective and hence is an order isomorphism. (5) $\Rightarrow$ (4). Let $0 \neq r \in R$ be a nonunit. Let $P_1, \dots, P_n$ be the height-one prime ideals containing (r) . Now, since the natural map $G(R) \rightarrow \bigoplus_{P \in X^{(1)}} G(R_P)$ is surjective, for each $i = 1, \dots, n$, there exists a $k_i \in K$ with $k_i U(R_P) = rU(R_{P_i})$ and $k_i U(R_P) = 1U(R_P)$ for each $P \in X^{(1)} - \{P_i\}$. Note that each $k_i \in \bigcap_{P \in X^{(1)}} R_P = R$. Moreover, since P_i is the only height-one prime ideal containing (k_i) , $(k_i) = \bigcap_{P \in X^{(1)}} (k_i R_P) = k_i R_{P_i} \cap R$ is P_i -primary. Now $rU(R_P) = k_1 \cdots k_n U(R_P)$ for each $P \in X^{(1)}$, so $r = uk_1 \cdots k_n$, where u is a unit of R . (4) $\Rightarrow$ (7). By [2, Corollary 14], $R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite. Let A be a t -invertible ideal of R . We must show that A_t is principal. We may assume that A is an integral ideal. We have already observed that every maximal t -ideal of R has height one. Hence, by a result of Griffin [7, Proposition 4], A_P is principal for each $P \in X^{(1)}$. Moreover, since the intersection is locally finite, $A_P = R_P$ except, say, for $P_1, \dots, P_n \in X^{(1)}$. But at P_i , $A_{P_i} = q_i R_{P_i}$, where (q_i) is P_i -primary. So $A \subseteq (q_i)_{P_i} \cap R = (q_i)$ and hence $A \subseteq (q_1) \cap \cdots \cap (q_n) = (q_1 \cdots q_n)$. Thus $A = q_1 \cdots q_n A'$, where A' is contained in no height-one prime ideals. Hence $A'_t = R$. So $A_t = (q_1 \cdots q_n A')_t = (q_1 \cdots q_n A'_t)_t = (q_1 \cdots q_n)_t = (q_1 \cdots q_n)$ is principal. (7) $\Rightarrow$ (4). Let $0 \neq x \in R$ be a nonunit. Let $P_1, \dots, P_n$ be the height-one prime ideals containing (x) . Let $Q_i = (x)_{P_i} \cap R$, so $(x) = Q_1 \cap \cdots \cap Q_n$, where Q_i is P_i -primary. Now $x^{-1}Q_1 \cdots Q_n$ is an ideal of R and $x^{-1}Q_1 \cdots Q_n R_P = R_P$ for each height-one prime ideal of R . Hence, by [1, Theorem 1], $(x^{-1}Q_1 \cdots Q_n)^* = R$ where $*$ is the finite character star operation given by $A^* = \bigcap_{P \in X^{(1)}} AR_P$. Since $*$ has finite character, $R = (x^{-1}Q_1 \cdots Q_n)^* \subseteq (x^{-1}Q_1 \cdots Q_n)_t \subseteq R$, so $(x^{-1}Q_1 \cdots Q_n)_t = R$. Thus each Q_i is t -invertible and hence $Q_{it} = (q_i)$ for some $q_i \in R$. Also, $(x) = (Q_1 \cdots Q_n)_t = (Q_{1t} \cdots Q_{nt})_t = ((q_1)) \cdots ((q_n))_t = (q_1) \cdots (q_n)$. Now $(q_i) \supseteq Q_i$ and hence $(q_i)_P = R_P$ except possibly at $P = P_i$. So $Q_i = (x)_{P_i} \cap R = (q_i)_{P_i} \cap R = q_i R_{P_i} \cap (\bigcap_{P \in X^{(1)} - \{P_i\}} q_i R_P) = (q_i)$. Hence (q_i) is P_i -primary, so x is a product of primary elements. $\square$

Corollary 1. *Let R be a Noetherian integral domain. Then R is weakly factorial if and only if every grade-one prime ideal has height one and $Cl_t(R) = 0$.*

Proof. $(\Rightarrow)$. A grade-one prime ideal is contained in a maximal t -ideal and hence has height one. By the previous theorem, $Cl_t(R) = 0$. $(\Leftarrow)$. If every grade-one prime ideal has height one, $R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite [8, Theorem 53]. By the previous theorem, R is weakly factorial. $\square$

In the spirit of Mott [9], we use the Theorem and the Krull–Kaplansky–Jaffard–Ohn Theorem to obtain a result of A. Bigard (see, for example, [3, Theoreme 14.4.3, pp. 290–291]) concerning lattice ordered groups.

Corollary 2 (A. Bigard). *A lattice ordered abelian group G is a cardinal sum of Archimedean totally ordered groups if and only if every convex directed subgroup of G is a cardinal summand of G .*

Proof. Let G be a lattice ordered abelian group. By the Krull–Kaplansky–Jaffard–Ohm Theorem ([6, Theorem 18.6] or [9, Theorem 2.1]), G is order isomorphic to the group of divisibility of a Bézout domain R . By the Theorem, $G(R)$, and hence G , has the property that every convex directed subgroup is a cardinal summand if and only if R is weakly factorial. But by [2, Theorem 21] a Bézout domain R is weakly factorial if and only if $G(R)$ is order isomorphic to a cardinal sum of rank one (equivalently, Archimedean) totally ordered abelian groups. $\square$

As previously stated, examples of weakly factorial domains include UFD's and one-dimensional semi-quasi-local domains. Actually, it is easily seen that a one-dimensional domain R is weakly factorial if and only if R is Laskerian (i.e., every ideal has a primary decomposition) and $\text{Pic}(R) = 0$. Also, an integral domain that is both a GCD-domain and a generalized Krull domain is weakly factorial. (An integral domain R is called a *generalized Krull domain* if $R = \bigcap V_\lambda$, where each V_λ is an essential rank one valuation overring of R and the intersection is locally finite.) In fact, according to [2, Theorem 20], the following conditions on an integral domain R are equivalent: (1) R is a weakly factorial GCD-domain; (2) R is weakly factorial generalized Krull domain; and (3) R is a generalized Krull domain and a GCD-domain. To this list may be added (4) R is weakly factorial and, for each height-one prime ideal P of R (equivalently, maximal t -ideal P of R), R_P is a valuation domain and (5) R is weakly factorial and if p and q are non v -coprime primary elements of R , then $p|q$ or $q|p$. (Use [2, Theorem 18] and its proof). We next give another source of weakly factorial domains.

Example. Let $K \subseteq L$ be fields. Then $R = K + XL[X] = \{f(X) \in L[X] | f(0) \in K\}$ is an atomic weakly factorial domain. It is easily seen that every nonzero nonunit of R may be written uniquely in one of the following two forms: $p_1 \cdots p_n$ or $p_1 \cdots p_n(aX^r)$, where $p_1, \dots, p_n$ are principal primes of R and aX^r ($r \geq 1$, $a \in L$) is primary.

It is well known [8, Theorem 5] that an integral domain R is a UFD if and only if every nonzero prime ideal contains a nonzero principal prime ideal. The proof is based on the fact that the set of principal primes forms a saturated multiplicatively closed set. A similar result for weakly factorial domains involving primary elements is not true.

For let R be a Dedekind domain that is not a PID, but has torsion class group. Since some power of each prime ideal is principal, each prime ideal contains a principal primary ideal, but R is not weakly factorial. In this case, the multiplicatively closed set of all products of primary elements is not saturated. As a concrete example, suppose that R has nonprincipal maximal ideals M and N with $M^2 = (x)$, $N^2 = (y)$, and $MN = (z)$. Then $(z^2) = (MN)^2 = M^2N^2 = (xy)$. So z^2 is a product of primary elements, but z itself is not. Note that x , y , and z are all irreducible. Thus a single irreducible element of a Krull domain may be primary without being prime. However, a Krull domain is a UFD if and only if each irreducible element is primary.

However, for any integral domain R and any nonzero prime ideal P of R , it is easily proved that $S_P = \{\lambda p_1 \cdots p_n \mid \lambda \in U(R), n \geq 0, (p_i) \text{ is } P\text{-primary}\}$ is a saturated multiplicatively closed subset of R . Let S be the multiplicatively closed subset of R consisting of all products of primary elements. Then, in $G(R)$, $\langle S \rangle = \oplus \langle S_P \rangle$ where the direct sum runs over all nonzero primes P of R with P containing a P -primary element. Even though each $\langle S_P \rangle$ is a convex directed subgroup of $G(R)$, $\oplus \langle S_P \rangle$ need not be convex, in fact, $\oplus \langle S_P \rangle$ is convex if and only if S is saturated. Returning to the case where R is a Dedekind domain with torsion class group but is not a PID, we see that each $\langle S_P \rangle$ is order isomorphic to $(\mathbb{Z}, +)$. It is interesting to note that in this case we have $\bigoplus_{P \in X^{(1)}} \langle S_P \rangle \subsetneq G(R) \subsetneq \bigoplus_{P \in X^{(1)}} G(R_P)$, with $\bigoplus_{P \in X^{(1)}} \langle S_P \rangle$ being a directed lattice ordered subgroup of $G(R)$ and $G(R)$ being a directed lattice ordered subgroup of $\bigoplus_{P \in X^{(1)}} G(R_P)$. Moreover, all three groups are (algebraically) isomorphic to a direct sum of $|X^{(1)}|$ copies of $(\mathbb{Z}, +)$.

REFERENCES

1. D. D. Anderson, *Star-operations induced by overrings*, Comm. Alg. **16** (1988), 2535–2553.
2. D. D. Anderson and L. A. Mahaney, *On primary factorizations*, J. Pure Appl. Alg. **54** (1988), 141–154.
3. A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et anneaux réticulés*, Lecture Notes in Math., vol. 608, Springer-Verlag, New York, 1977.
4. A. Bouvier, *Le groupe des classes d'un anneau intégré*, 107^e Congrès national des Sociétés savantes, Brest, 1982, sciences, fasc. IV, 85–92.
5. A. Bouvier and M. Zafrullah, *On some class groups of an integral domain*, Bull. Soc. Math. Grèce (to appear).
6. R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972.
7. M. Griffin, *Some results on v -multiplication rings*, Canad. J. Math. **19** (1967), 710–722.
8. I. Kaplansky, *Commutative rings*, revised ed., The University of Chicago Press, Chicago and London, 1974.

9. J. L. Mott, *The group of divisibility and its application*, Lecture Notes in Math., vol. 311, Springer-Verlag, New York, 1972, 194–208.
10. —, *Convex directed subgroups of a group of divisibility*, Canad. J. Math. **26** (1974), 532–542.
11. J. L. Mott and M. Schexnayder, *Exact sequences of semi-value groups*, J. Reine Angew. Math. **283/284** (1976), 388–401.

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