THE PRODUCT OF A LINDELÖF SPACE WITH THE SPACE OF IRRATIONALS UNDER MARTIN'S AXIOM

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ABSTRACT. In this paper, we give an example of a Lindelöf space whose product with the space of irrationals is not Lindelöf provided that Martin's axiom holds. Our result is an improvement of a Michael's construction using the Continuum Hypothesis.

In 1971, E. Michael constructed a Lindelöf space whose product with the space of irrationals is not Lindelöf provided that the Continuum Hypothesis holds. We have improved Michael's result by showing that the Continuum Hypothesis can be relaxed to Martin's axiom (MA). For a related result, see [BD]. Some positive results are presented. Finally, we formulate certain set-theoretical statements that ensure the existence of our example.

The topological terminology follows [E], and we have adopted set-theoretical terminology from [K]. In our paper, N stands for the set of natural numbers and P for the space of irrational numbers. The irrational numbers will be viewed as sequences of natural numbers. The symbol c will denote the initial ordinal number of the cardinality continuum. Other ordinal numbers will be denoted by Greek letters; in particular, ω will stand for the first infinite ordinal number and ω_1 for the first uncountable ordinal number. If $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ are irrational numbers, we say that $x <_* y$ ($x \leq_* y$) if there is $m \in N$ such that $x(j) < y(j)$ ($x(j) \leq y(j)$) for $m < j$.

$S \subset P$ is called an α -scale if and only if $S = \{x_\beta : \beta < \alpha\}$, where $x_{\beta_1} <_* x_{\beta_2}$ whenever $\beta_1 < \beta_2 < \alpha$, and if $x \in P$, then there is $\beta < \alpha$ so that $x <_* x_\beta$.

The symbol $cf(\alpha)$ will stand for the cofinality of α . If $x_\alpha \in P$, then X_α will be defined by $X_\alpha = \{x \in P : x \leq_* x_\alpha\}$.

Example (MA). *There is a Lindelöf space X such that the Cartesian product $X \times P$ is not Lindelöf.*

Construction. MA implies that there is a c -scale $S = \{x_\alpha : \alpha < c\}$ in P , see [K], exercise 8, p. 87. Let (Y, d) be a metric compactification of P . The

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space X as a set is equal to Y . To simplify notation, we shall denote X by X_c . If $\alpha \leq c$ and $x \in X_\alpha \setminus (\bigcup\{X_\beta: \beta < \alpha\})$, the base at x consists of all sets of the form $(X_\alpha \cap U) \setminus X_\beta$, where $x \in U$, U is open in (Y, d) , and $\beta < \alpha$.

Lemma 1(MA). *If $X'_\alpha = X_\alpha \setminus \bigcup\{X_\beta: \beta < \alpha\}$ and U is open in X containing X'_α , there is a sequence $(\gamma_i)_{i=1}^\infty$ in α such that $X_\alpha \setminus U \subset \bigcup_{i=1}^\infty X_{\gamma_i}$.*

Proof. If $cf(\alpha) = \omega$ it is trivial, so let us assume that $\omega < cf(\alpha)$. Because X'_α is Lindelöf, it is actually a metric, separable subspace of (Y, d) , since $\omega < cf(\alpha)$, we can assume without loss of generality that there is an open set H in (Y, d) and $\beta_1 < \alpha$ so that $U = (H \cap X_\alpha) \setminus X_{\beta_1}$. If $\alpha = c$, $X_c \setminus H$ is a compact subset of P . Hence, there is $y = (y(n))_{n=1}^\infty$ so that

$$X_c \setminus H \subset \{x \in P: x(n) \leq y(n) \text{ for } n \in N\} = K.$$

Since $S = \{x_\beta: \beta < c\}$ is a c -scale, there is $\beta_2 < c$ such that $y <_* x_{\beta_2}$, which implies that $K \subset X_{\beta_2}$ and consequently $X_c \setminus U \subset X_{\beta_1} \cup X_{\beta_2}$.

Let us assume that $\alpha < c$. Observe that X_γ is σ -compact with respect to the topology of P for $\gamma < c$; therefore, $X_\gamma = \bigcup\{X_\gamma(n): n \in N\}$, where $X_\gamma(n)$ is compact in P . Then:

$$X_\alpha \setminus H \subset \bigcup\{X_\gamma: \gamma < \alpha\} = \bigcup\{X_\gamma(m): m \in N \text{ and } \gamma < \alpha\}$$

and

$$D_n = X_\alpha(n) \setminus H = \bigcup\{(X_\alpha(n) \setminus H) \cap X_\gamma(m): \gamma < \alpha \text{ and } m \in N\}$$

for $n \in N$. As D_n is a compact subspace of P , which is a union of less than continuum compact subsets, the conclusion of Lemma 1 will follow from Lemma 2.

Lemma 2(MA). *If Z is a compact metric space in which $Z = \bigcup\{Z_\gamma: \gamma < \beta\}$, where $\beta < c$ and Z_γ is compact for $\gamma < \beta$, then there is a sequence $(\gamma_i)_{i=1}^\infty$ in β so that $\bigcup\{Z_{\gamma_i}: i \in N\} = Z$.*

Proof. Suppose that this is not the case. MA implies that

$$H_0 = \bigcup\{\text{Int}_Z Z_\gamma: \gamma < \beta\}$$

is not empty; otherwise, Z would be a union of less than continuum, closed, nowhere-dense subsets. Since Z is a hereditarily Lindelöf space, there is a countable subset $T(H_0)$ of β so that $H_0 \subset \bigcup\{Z_\gamma: \gamma \in T(H_0)\}$. Let us assume that for every γ less than δ where δ is a countable ordinal number, we have defined an open subset H_γ of Z and a countable subset $T(H_\gamma)$ of β so that $H_\gamma \subset \bigcup\{Z_{\gamma'}: \gamma' \in T(H_\gamma)\}$ and $H_\gamma \setminus (\bigcup\{H_{\gamma'}: \gamma' < \gamma\}) \neq \emptyset$. Put $H = \bigcup\{H_\gamma: \gamma < \delta\}$ and $A = Z \setminus H$. A is not empty; otherwise, the conclusion of Lemma 2 holds. Put $H'_\delta = \bigcup\{\text{Int}_A(Z_\gamma \setminus H): \gamma < \beta\}$. MA implies that H'_δ is not empty. Because H'_δ is Lindelöf, there is a countable subset $T(H'_\delta)$ of β so that $H'_\delta = \bigcup\{\text{Int}_A(Z_\gamma \setminus H): \gamma \in T(H'_\delta)\}$. Put $H_\delta = H'_\delta \cup H$. Because the complement

of H_δ is closed, we infer that H_δ is open. Put $T(H_\delta) = T(H'_\delta) \cup \bigcup \{T(H_\gamma) : \gamma < \delta\}$; then $H_\delta \subset \bigcup \{T_\gamma : \gamma \in T(H_\delta)\}$ and $\emptyset \neq H'_\delta = H_\delta \setminus \bigcup \{H_\gamma : \gamma < \delta\}$. By the construction, $\{H_\gamma : \gamma < \omega_1\}$ is a strictly increasing sequence of open sets in a separable metric space, and thus is a contradiction.

Lemma 3(MA). X_α is a Lindelöf space for every $\alpha \leq c$.

Proof. Observe that X_0 is metric and separable. Let us assume that X_α is Lindelöf for $\alpha < \beta$. Let $\mathcal{U} = \{U_s : s \in S\}$ be an open cover of X_β . Because X'_β is Lindelöf, there is a countable set S' in S so that $X'_\beta \subset \bigcup \{U_s : s \in S'\} = U$. By Lemma 1, there is $(\gamma_i)_{i=1}^\infty$ in β so that $X_\beta \setminus U \subset \bigcup \{X_{\gamma_i} : i \in N\}$. By the inductive assumption, there is a countable subset S_i , of S , so that $X_{\gamma_i} = \bigcup \{X_{\gamma_i} \cap U_s : s \in S_i\}$ for $i \in N$. Therefore, $X_\beta = \bigcup \{U_s : s \in \bigcup \{S_i \cup S' : i \in N\}\}$.

Lemma 4. The product $X \times P$ is not Lindelöf.

Proof. As $K = \{(p, p) : p \in P\}$ is a closed subset of $X \times P$ and the family $\{K \cap (X_\alpha \times P) : \alpha < c\}$ is a cover of K without countable subcover, we infer that $X \times P$ is not Lindelöf.

Let (S) be the following statement: For every compact subset Z of P , and for every strictly increasing sequence $\{x_\gamma : \gamma < \alpha\}$ with respect to $<_*$ in Z where $\omega < cf(\alpha)$, there is $x \in Z$ so that $x_\gamma \leq_* x$ for $\gamma < \alpha$.

I was not able to prove or disprove (S). Still, it is possible to demonstrate the following proposition.

Proposition. If (S) holds, there is a Lindelöf space X in which the product $X \times P$ is not Lindelöf.

Proof. Let (Y, d) be a metric compactification of P . There is an uncountable, strictly increasing, and unbounded sequence $(x_\alpha)_{\alpha < \omega_\beta}$ in P with respect to $<_*$. Put $C_{\omega_\beta} = Y \setminus P$ and $C_\alpha = \{x \in P : x \leq_* x_\alpha \text{ and } x_\gamma <_* x \text{ for } \gamma < \alpha\}$ for $\alpha < \omega_\beta$. Observe that $C_{\alpha_1} \cap C_{\alpha_2} = \emptyset$ whenever $\alpha_1 \neq \alpha_2$. Put $X = \bigcup \{C_\alpha : \alpha \leq \omega_\beta\}$. If $\alpha \leq \omega_\beta$ and $x \in C_\alpha$, then the base at x consists of the sets containing x of the form

$$U = H \cap \left(\bigcup \{C_\gamma : \gamma_1 < \gamma \leq \alpha\} \right),$$

where H is open in (Y, d) and $\gamma_1 < \alpha$. Since the topology of C_α coincides with the topology generated by d , we conclude that C_α is Lindelöf. In order to show that X is Lindelöf, it is enough to prove Lemma 5.

Lemma 5 [S]. If $\alpha \leq \omega_\beta$ and U is an open subset of X containing C_α , there is a sequence $(\gamma_n)_{n=1}^\infty$ in α such that:

$$\bigcup \{C_\gamma : \gamma \leq \alpha\} \setminus U \subset \bigcup \{C_\gamma : \gamma \leq \gamma_n \text{ and } n \in N\}.$$

Observe that Lemma 5 is similar to Lemma 1.

Proof. If $cf(\alpha) = \omega$ it is trivial, so let us assume that $\omega < cf(\alpha)$. Observe that $\omega < cf(\omega_\beta)$; otherwise, $(x_\alpha)_{\alpha < \omega_\beta}$ is bounded. As C_α is Lindelöf and

$\omega < cf(\omega_\beta)$, we can assume without loss of generality that there are $\gamma'_1 < \alpha$ and an open subset H of (Y, d) so that $U = H \cap \left(\bigcup \{C_\gamma: \gamma'_1 < \gamma \leq \alpha\} \right)$. Suppose that $\alpha = \omega_\beta$. Because Y is compact, we infer that $K = Y \setminus H$ is a compact subset of P . Hence, there is y in P so that $y' \leq y$ for $y' \in K$. Since $(x_\gamma)_{\gamma < \omega_\beta}$ is unbounded, we conclude that there is $\gamma'_1 < \gamma_2 < \omega_\beta$ so that $x_{\gamma_2} \not\leq_* y$. Observe that if $\gamma_2 < \gamma$ and $x \in C_\gamma$, then $x_{\gamma_2} <_* x$ so that $x \not\leq_* y$. From this observation, it follows that $\bigcup \{C_\gamma: \gamma_2 \leq \gamma \leq \omega_\beta\} \subset H$; therefore, by the definition of U , $\bigcup \{C_\gamma: \gamma_2 < \gamma \leq \omega_\beta\} \subset U$.

Let us assume that $\alpha < \omega_\beta$. Observe that $\bigcup \{C_\gamma: \gamma \leq \alpha\} \subset X_\alpha$ and that $A_n = X_\alpha(n) \setminus H$ is compact with respect to the topology of P . If $\Gamma_n = \{\gamma < \alpha: C_\gamma \cap A_n \neq \emptyset\}$ is not empty, then (S) implies that $\sup \Gamma_n = \gamma_n < \alpha$. Suppose that this is not the case, and choose $x'_\gamma \in X_\gamma \cap A_n$ for $\gamma \in \Gamma_n$. Observe that $\{x'_\gamma: \gamma \in \Gamma_n\}$ is a strictly increasing sequence in A_n of uncountable cofinality. By (S) there is $x \in A_n$ so that $x'_\gamma \leq_* x$ for $\gamma \in \Gamma_n$. Since $\sup \Gamma = \alpha$ and $x \leq x_\alpha$, we infer that $x \in C_\alpha \subset U$, contradiction. We conclude that:

$$\bigcup \{C_\gamma: \gamma \leq \alpha\} \setminus U \subset \bigcup \{C_\gamma: \gamma \leq \gamma_n \text{ and } n \in N\} \cup \bigcup \{C_\gamma: \gamma \leq \gamma'_1\}$$

Remark. Observe that in order to construct the last example, we needed a slightly weaker statement than (S), (S'), which says that there is an unbounded, strictly increasing sequence $(x_\gamma)_{\gamma < \omega_\beta}$ in P such that for every $\alpha < \omega_\beta$ of uncountable cofinality and every compact subset Z of P , if $C_\gamma = \{x \in P: x \leq_* x_\gamma\}$ and $x_{\gamma'} <_* x$ for $\gamma' < \gamma$, $\Gamma = \{\gamma < \alpha: C_\gamma \cap Z \neq \emptyset\}$, $\sup \Gamma = \alpha$, and $x'_\gamma \in C_\gamma \cap Z$ for $\gamma \in \Gamma$, then $x \in Z$ will satisfy $x'_\gamma \leq x$ for $\gamma \in \Gamma$.

Observe that (S') holds if there is an unbounded, strictly increasing sequence of length ω_1 . Therefore, our construction improves the result of Burke and Davis [BD], which says that a ω_1 -scale implies the existence of a Lindelöf space whose product with the space of irrationals is not Lindelöf.

We say that an open cover $\mathcal{U} = \{U_\gamma: \gamma < \alpha\}$ of X is essential if $\omega < cf(\alpha)$ and $U_\gamma \setminus \bigcup \{U_{\gamma'}: \gamma' < \gamma\}$ is not empty for $\gamma < \alpha$.

Theorem (MA). *If X is a Lindelöf space, then there is no essential open cover $\mathcal{U} = \{U_\gamma: \gamma < \alpha\}$ of the product $X \times P$ so that $cf(\alpha) < c$.*

Proof. If this is not the case, there is an essential cover $\mathcal{U} = \{U_\gamma: \gamma < \alpha\}$ of $X \times P$ so that $cf(\alpha) < c$. Let p_γ be an element of P for $\gamma < \alpha$ so that $X \times \{p_\gamma\} \setminus \bigcup \{U_{\gamma'}: \gamma' \leq \gamma\}$ is not empty. Let $\{\gamma_\beta: \beta < cf(\alpha)\}$ be a cofinal subset of α . MA implies that there is $p \in P$ so that $p_{\gamma_\beta} <_* p$ for $\beta < cf(\alpha)$, see [K], exercise 8, p. 87. Since $\omega < cf(\alpha)$, $\{p_{\gamma_\beta}: \beta < cf(\alpha)\} \subset Z = \{x \in P: x \leq_* p\}$, Z is σ -compact, and the product of a σ -compact space and a Lindelöf space

is Lindelöf, we infer that there is $\gamma' < \alpha$ so that $X \times \{p_{\gamma_\beta} : \beta < cf(\alpha)\} \subset \bigcup \{U_\gamma : \gamma \leq \gamma'\}$. On the other hand, if β is such that $\gamma' < \gamma_\beta$, then by the definition of p_{γ_β} , $X \times \{p_{\gamma_\beta}\}$ is not a subset of $\bigcup \{U_\gamma : \gamma \leq \gamma'\}$, contradiction.

Corollary 1 (MA and $\omega_1 < c$). *If X is Lindelöf, then the product $X \times P$ does not contain an uncountable discrete subset.*

Corollary 2 (MA and $\omega_1 < c$). *If X is a regular space, the following conditions are equivalent:*

- (a) $X \times P$ is Lindelöf,
- (b) $X \times P$ is meta-Lindelöf.

If $\omega_n < c$ also holds for $n \in N$, (a) and (b) are equivalent to:

- (c) *for every open cover $\mathcal{U}$, there is a natural number n and a refinement $\mathcal{V}$ so that every point of $X \times P$ belongs to at most ω_n elements of $\mathcal{V}$.*

Corollary 3 (MA and $\omega_\omega < c$). *If X is a Lindelöf space of weight less than ω_ω , then the product $X \times P$ is Lindelöf.*

We shall finish with the following problems:

Problem 1. *Does (S) hold in ZFC?*

Problem 2. *Does (S') hold in ZFC?*

Problem 3. *Let us assume that X is a Lindelöf space and that MA holds. Does every open cover of $X \times P$ with cardinality less than c have a countable subcover?*

Observe that a positive answer to the first problem implies a positive answer to the second, but that (S') is sufficient to construct a real example of a Lindelöf space whose product with the space of irrationals is not Lindelöf.

Added in proof. Recently D. H. Fremlin has proved that (S) does not hold provided that $b = c$.

REFERENCES

- [BD] D. K. Burke and S. W. Davis, *Subsets of ω_ω and generalized metric spaces*, Pacific J. Math. **110** (1984), 273–281.
- [E] R. Engelking, *General topology*, Warszawa, 1977.
- [K] K. Kunen, *Set theory, An introduction to independence proofs*, Stud. Logic Foundations Math. vol. 102.
- [M] E. Michael, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Compositio Math. **23** (1971), 199–214.

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