NONGEOMETRIC CONVERGENCE OF BEST L_p ($p \neq 2$) POLYNOMIAL APPROXIMANTS

K. G. IVANOV AND E. B. SAFF

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ABSTRACT. For an arbitrary function f analytic in the disk $D: |z| < 1$ and continuous in $\bar{D}$, we show that geometric convergence in D of best L_p ($1 \leq p \leq \infty$) polynomial approximants to f on $C: |z| = 1$ is assured only when $p = 2$.

1. INTRODUCTION

Let $\mathcal{A}$ denote the algebra of functions f analytic in the open unit disk $D: |z| < 1$ and continuous on $\bar{D}: |z| \leq 1$. It is well known that for every $f \in \mathcal{A}$ the partial sums

$$s_n(z) = s_n(f, z) := \sum_{k=0}^n a_k z^k$$

of the Maclaurin expansion $\sum_0^\infty a_k z^k$ of f converge geometrically fast on each closed subset of D ; indeed, for $z \in D$,

$$\limsup_{n \rightarrow \infty} |f(z) - s_n(z)|^{1/n} \leq |z| < 1.$$

Since the polynomials s_n are also least squares (best L_2 approximants) to f on the unit circle $C: |z| = 1$; that is,

$$\int_C |f(z) - s_n(z)|^2 |dz| \leq \int_C |f(z) - P_n(z)|^2 |dz|, \quad \forall P_n \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ denotes the collection of all polynomials of degree at most n , it is natural to ask, more generally, if best L_p ($1 \leq p \leq \infty$) polynomial approximants to f on C also enjoy the property of geometric convergence in D . The

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purpose of this paper is to show that, surprisingly, $p = 2$ is the only value for which geometric convergence holds for all $f \in \mathcal{A}$.

For each $1 \leq p \leq \infty$, we let $\|\cdot\|_p$ denote the L_p -norm on C ; that is

$$(1) \quad \|g\|_p := \left(\frac{1}{2\pi} \int_{|z|=1} |g(z)|^p |dz| \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$(2) \quad \|g\|_\infty := \max_{|z|=1} |g(z)|.$$

By $Q_{n,p}^* = Q_{n,p}^*(f)$ we denote the unique best L_p approximant to f out of $\mathcal{P}_n$ in the sense that

$$(3) \quad E_{n,p}(f) := \|f - Q_{n,p}^*(f)\|_p \leq \|f - Q_n\|_p, \quad \forall Q_n \in \mathcal{P}_n.$$

It is easy to verify (cf. [2, §5.8]) that, for every $f \in \mathcal{A}$,

$$(4) \quad \lim_{n \rightarrow \infty} Q_{n,p}^*(f, z) = f(z), \quad \text{for } z \in D.$$

Our goal is to show that for $p \neq 2$ the convergence in (4) may not be geometric. We shall prove

Theorem 1.1. *For each $p \neq 2$, $1 \leq p \leq \infty$, there exists an $F \in \mathcal{A}$ such that $F(0) = 0$ but*

$$(5) \quad \limsup_{n \rightarrow \infty} |Q_{n,p}^*(F, 0)|^{1/n} = 1.$$

It is straightforward to show that if f is analytic on the closed disk $|z| \leq 1$, then

$$\limsup_{n \rightarrow \infty} |f(z) - Q_{n,p}^*(z)|^{1/n} < 1, \quad z \in D.$$

Hence the function F of Theorem 1.1 must have a singularity on C ; however, our construction shows that F can be arbitrarily smooth on C .

2. CONSTRUCTION OF F

The proof of Theorem 1.1 requires some simple lemmas.

Lemma 2.1. *Let $g \in \mathcal{A}$ and set $g_n(z) := g(z^{n+1})$. Then for any $Q \in \mathcal{P}_n$,*

$$(6) \quad \|g - Q(0)\|_p = \|g_n - Q(0)\|_p \leq \|g_n - Q\|_p.$$

Proof. The equality in (6) is obvious. The inequality follows from the fact that

$$g_n(\omega^k z) = g_n(z), \quad \omega := e^{2\pi i/(n+1)}, \quad k = 0, 1, \dots,$$

and the identity

$$Q(0) = \frac{1}{n+1} \sum_{k=1}^{n+1} Q(\omega^k z), \quad \text{for } Q \in \mathcal{P}_n.$$

Indeed,

$$\begin{aligned}\|g_n - Q(0)\|_p &= \left\| \frac{1}{n+1} \sum_{k=1}^{n+1} (g_n(z) - Q(\omega^k z)) \right\|_p \\ &= \frac{1}{n+1} \left\| \sum_{k=1}^{n+1} (g_n(\omega^k z) - Q(\omega^k z)) \right\|_p \\ &\leq \frac{1}{n+1} \sum_{k=1}^{n+1} \|g_n(\omega^k z) - Q(\omega^k z)\|_p \\ &= \|g_n - Q\|_p. \quad \square\end{aligned}$$

Corollary 2.2. Let g and g_n be as in Lemma 2.1. Then

$$(7) \quad E_{n,p}(g_n) = E_{0,p}(g), \quad n \geq 0,$$

where $E_{n,p}$ is as defined in (3).

Proof. Using (6), we get

$$E_{0,p}(g) = E_{0,p}(g_n) \geq E_{n,p}(g_n)$$

and

$$\begin{aligned}E_{n,p}(g_n) &= \|g_n - Q_{n,p}^*(g_n)\|_p \\ &\geq \|g_n - Q_{n,p}^*(g_n, 0)\|_p \\ &= \|g - Q_{n,p}^*(g_n, 0)\|_p \geq E_{0,p}(g). \quad \square\end{aligned}$$

Lemma 2.3. Let $1 \leq p \leq \infty$, $p \neq 2$. Then there exists a $g \in \mathcal{P}_2$ such that $\|g\|_p = 1$, $g(0) = 0$, but $\lambda := Q_{0,p}^*(g) \neq 0$. Moreover, there is an increasing continuous function φ on $[0, \infty)$ such that $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$, $\varphi(x) \rightarrow \infty$ as $x \rightarrow \infty$, and

$$(8) \quad \|g - \lambda\|_p + \varphi(|\alpha - \lambda|) \leq \|g - \alpha\|_p, \quad \forall \alpha \in \mathbb{C}.$$

Proof. Assume first that $p < \infty$ and consider the integral

$$(9) \quad I(r, s) := \int_0^{2\pi} (1 + 2s \cos \theta + s^2)^r (\cos \theta + s \cos 2\theta) d\theta,$$

where $r \in \mathbb{R}$ and $0 \leq s < 1$. We have

$$\begin{aligned}I(r, 0) &= \int_0^{2\pi} \cos \theta d\theta = 0, \\ \frac{\partial I}{\partial s}(r, s) &= \int_0^{2\pi} (1 + 2s \cos \theta + s^2)^{r-1} \{r(2 \cos \theta + 2s)(\cos \theta + s \cos 2\theta) \\ &\quad + (1 + 2s \cos \theta + s^2) \cos 2\theta\} d\theta, \\ \frac{\partial I}{\partial s}(r, 0) &= \int_0^{2\pi} \{2r \cos^2 \theta + \cos 2\theta\} d\theta = 2\pi r.\end{aligned}$$

Therefore, $I(r, s) \neq 0$ for $s > 0$ sufficiently small, provided $r \neq 0$. For

$$r = \frac{p}{2} - 1 \neq 0,$$

fix such an s and set

$$(10) \quad G(z) := z + sz^2.$$

Let $c = a + ib$, $a, b \in \mathbf{R}$, be any complex constant. Then

$$(11) \quad \|G - c\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} \{(\cos \theta + s \cos 2\theta - a)^2 + (\sin \theta + s \sin 2\theta - b)^2\}^{p/2} d\theta.$$

Let $\Lambda := Q_{0,p}^*(G)$. Since $\|G - \Lambda\|_p^p \leq \|G - c\|_p^p$ for all $c \in \mathbf{C}$, we have

$$(12) \quad \frac{\partial}{\partial a} \|G - c\|_p^p|_{c=\Lambda} = 0.$$

If $\Lambda = 0$, then (12) becomes

$$-\frac{p}{2\pi} \int_0^{2\pi} (1 + 2s \cos \theta + s^2)^{p/2-1} (\cos \theta + s \cos 2\theta) d\theta = 0,$$

which contradicts the fact that $I(p/2 - 1, s) \neq 0$. Therefore, $\Lambda \neq 0$, and so $g := G/\|G\|_p$ gives the desired function for the case $1 \leq p < \infty$, $p \neq 2$.

For $p = \infty$, it is straightforward to verify that for

$$(13) \quad G(z) = z + \frac{1}{2}z^2,$$

we have

$$E_{0,\infty}(G) = \frac{3\sqrt{3}}{4} = \left\| G - \frac{1}{4} \right\|_\infty = \left| G(e^{i\pi/3}) - \frac{1}{4} \right|,$$

so that $\Lambda := Q_{0,\infty}^*(G) = 1/4 \neq 0$. Again we set $g := G/\|G\|_\infty$.

For the second part of the lemma, we let

$$\varphi(x) := \inf\{\|g - \lambda - \gamma\|_p : |\gamma| \geq x\} - \|g - \lambda\|_p.$$

Then the asserted properties of φ follow from the uniqueness of the best approximant. $\square$

Remark. Lemma 2.3 is the only place in the construction where we need $p \neq 2$. Of course, for $p = 2$, the condition $g(0) = 0$ implies that $\lambda = 0$.

We can now give the

Proof of Theorem 1.1. In the construction we shall use the following basic properties of the error $E_{n,p}(f)$, $1 \leq p \leq \infty$:

$$(14) \quad E_{n,p}(f_1 + f_2) \leq E_{n,p}(f_1) + E_{n,p}(f_2)$$

$$(15) \quad E_{n,p}(\alpha f) = |\alpha| E_{n,p}(f), \quad \alpha \in \mathbf{C},$$

$$(16) \quad E_{n,p}(f) \leq \|f\|_p,$$

$$(17) \quad |f(0) - Q_{n,p}^*(f, 0)| \leq \|f - Q_{n,p}^*(f)\|_1 \leq E_{n,p}(f),$$

where f, f_1, f_2 belong to $\mathcal{A}$. Let $1 \leq p \leq \infty$, $p \neq 2$, be fixed and let g, λ , and φ be as in Lemma 2.3. Choose $\delta > 0$ so that

$$(18) \quad \delta < |\lambda|/2$$

and

$$(19) \quad \varphi(x) < \delta \text{ implies } |x| \leq |\lambda|/4.$$

Then (18) and (16) imply that $\delta < 1$, since

$$|\lambda| \leq \|g - \lambda\|_p + \|g\|_p \leq 2\|g\|_p = 2.$$

Set $c_j := (\delta/5)^j$, $j = 0, 1, \dots$, and

$$(20) \quad F(z) := \sum_{j=0}^{\infty} c_j g(z^{n_j+1}) = \sum_{j=0}^{\infty} c_j g_{n_j}(z),$$

$$(21) \quad F_k(z) := F(z) - c_k g_{n_k}(z),$$

where the integers $\{n_j\}$ are defined inductively as follows: $n_0 := 0$, and n_k ($> n_{k-1}$) is chosen so large that

$$(22) \quad E_{n_k, p} \left(\sum_{j=0}^{k-1} c_j g_{n_j} \right) < \delta c_k / 4$$

and

$$(23) \quad n_k c_k \geq 1, \quad k > 0.$$

We note that for any function $f \in \mathcal{A}$, we have $E_{n,p}(f) \rightarrow 0$ as $n \rightarrow \infty$, so that (22) can always be satisfied. It is clear from the definitions that $F, F_k \in \mathcal{A}$ and $F(0) = F_k(0) = 0$.

In the estimates that follow, we shall suppress the subscript p . From (8), (6), and (21) we get

$$\begin{aligned} \mu_k &:= c_k \left\{ E_0(g) + \varphi \left(\left| \lambda - \frac{1}{c_k} [Q_{n_k}^*(F, 0) - Q_{n_k}^*(F_k, 0)] \right| \right) \right\} \\ &\leq c_k \left\| g - \frac{1}{c_k} [Q_{n_k}^*(F, 0) - Q_{n_k}^*(F_k, 0)] \right\| \\ &= \|c_k g_{n_k} - [Q_{n_k}^*(F, 0) - Q_{n_k}^*(F_k, 0)]\| \\ &\leq \|c_k g_{n_k} - Q_{n_k}^*(F) + Q_{n_k}^*(F_k)\| \\ &= \|[F - Q_{n_k}^*(F)] - [F_k - Q_{n_k}^*(F_k)]\| \\ &\leq E_{n_k}(F) + E_{n_k}(F_k). \end{aligned}$$

Thus from (14), (15), (7), and (22) it follows that

$$\begin{aligned}\mu_k &\leq E_{n_k}(c_k g_{n_k}) + 2E_{n_k}(F_k) \\ &\leq c_k E_0(g) + 2 \left\{ E_{n_k} \left(\sum_{j=0}^{k-1} c_j g_{n_j} \right) + E_{n_k} \left(\sum_{j=k+1}^{\infty} c_j g_{n_j} \right) \right\} \\ &\leq c_k E_0(g) + 2 \left\{ \frac{\delta c_k}{4} + \sum_{j=k+1}^{\infty} c_j \right\},\end{aligned}$$

where in the last inequality we used the fact that $\|g_{n_j}\| = 1$. Since $c_j = (\delta/5)^j$, we obtain

$$\begin{aligned}(24) \quad \mu_k &\leq c_k \left\{ E_0(g) + \frac{\delta}{2} + \frac{2\delta}{5} \left(\frac{1}{1-\delta/5} \right) \right\} \\ &< c_k \{E_0(g) + \delta\}.\end{aligned}$$

Therefore,

$$\varphi \left(\left| \lambda - \frac{1}{c_k} [Q_{n,k}^*(F, 0) - Q_{n_k}^*(F_k, 0)] \right| \right) < \delta,$$

and (19) gives

$$(25) \quad \left| \lambda - \frac{1}{c_k} [Q_{n_k}^*(F, 0) - Q_{n_k}^*(F_k, 0)] \right| < \frac{|\lambda|}{4}.$$

Now from (17) (with $f = F_k$, $n = n_k$), (21), (22), and (18), we get

$$\begin{aligned}(26) \quad |Q_{n_k}^*(F_k, 0)| &\leq E_{n_k}(F_k) \leq E_{n_k} \left(\sum_{j=0}^{k-1} c_j g_{n_j} \right) + E_{n_k} \left(\sum_{j=k+1}^{\infty} c_j g_{n_j} \right) \\ &\leq \frac{\delta c_k}{4} + \frac{c_k \delta}{5-\delta} \leq c_k \frac{\delta}{2} < c_k \frac{|\lambda|}{4}.\end{aligned}$$

Finally, (25) and (26) imply that

$$|Q_{n_k}^*(F, 0)| > \frac{|\lambda|}{2} c_k,$$

and so from (23) we see that (5) holds. $\square$

Some further results concerning the behavior of best L_∞ (uniform) polynomial approximants appear in [1].

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INSTITUTE FOR MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, SOFIA 1090, BULGARIA

INSTITUTE FOR CONSTRUCTIVE MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620