ON THE WEIGHTED INTEGRABILITY OF MAXIMAL ERGODIC RATIOS

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(Communicated by J. Marshall Ash)

ABSTRACT. Necessary and sufficient conditions on pairs of nonnegative weight functions are given which ensure that the maximal ergodic ratios defined by means of a measure preserving conservative ergodic transformation satisfy a reverse weak type inequality. From this, weighted integrability results may be deduced.

Let (Ω, Σ, μ) be a σ -finite measure space and let $T: \Omega \rightarrow \Omega$ be an ergodic, conservative measure preserving transformation. For fixed measurable $g > 0$ a.e., define the maximal ergodic ratio for nonnegative $f \in L^1(d\mu)$ and $x \in \Omega$ by

$$S(f, g) = \sup_{n \geq 1} \left(\frac{\sum_{j=0}^{n-1} f(T^j x)}{\sum_{j=0}^{n-1} g(T^j x)} \right)$$

and if $\lambda > 0$, set $A = \{x \in \Omega : S(f, g) > \lambda\}$. Derriennic [2] proved the reverse weak type inequality

$$(1) \quad \int_A f v \, d\mu \leq \lambda \int_{A \cup T^{-1}A} \omega \, d\mu, \quad \mu(A^c) > 0$$

for the (unweighted) case $v = 1$, $\omega = g$ and used this to deduce integrability properties of f from those of $S(f, g)$. The purpose of this note is to give conditions on the nonnegative weight pairs v, ω which ensure that (1) holds. Since (1) holds for all v, ω if Ω consists of a single atom, we assume that Ω contains at least two disjoint sets of positive measure. Then our conditions characterize the pairs v, ω satisfying (1) provided T satisfies the additional hypothesis $T^{-1}\Sigma = \Sigma$ (up to sets of measure zero). In particular this is the case if T is invertible. Sato [5] obtained a similar characterization of a closely related inequality for invertible T in the special case $g(x) = 1$.

Received by the editors November 10, 1988 and, in revised form, August 15, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 28D05.

Key words and phrases. Ergodic ratios, weighted inequalities, reverse weak type inequality, weighted integrability.

This research was supported in part by NSERC grant #A-8185.

Theorem. Suppose ω and v are nonnegative measurable weight functions on Ω . If

$$(2) \quad v(T^n x) \leq \frac{1}{2} \left(\frac{\sum_{j=0}^n \omega(T^j x)}{\sum_{j=0}^n g(T^j x)} \right)$$

for all $n \geq 1$ and almost all $x \in \Omega$, then (1) holds. Conversely, if $T^{-1}\Sigma = \Sigma$ and (1) holds, then

$$(3) \quad v(T^n x) \leq 2 \left(\frac{\sum_{j=0}^n \omega(T^j x)}{\sum_{j=0}^n g(T^j x)} \right)$$

for all $n \geq 1$ and almost all $x \in \Omega$.

Corollary. If (2) holds, then

$$\int_{\Omega} [S(f, g) + S(f, g) \circ T] \omega d\mu < \infty \quad \text{implies} \quad \int_{\Omega} [f \log^+(f/g)] v d\mu < \infty.$$

Proof. To prove that (2) implies (1), we first show that for each $k \geq 1$ and almost all $x \notin A$ we have

$$(4) \quad \sum_{i=1}^k f(T^i x) v(T^i x) \leq \lambda \sum_{i=0}^k \omega(T^i x).$$

To see this, fix $k \geq 1$ and for $x \in \Omega$ let $\{\tilde{v}_i\}_{i=1}^k$ be the least nonincreasing majorant of $\{v(T^i x)\}_{i=1}^k$, that is,

$$\tilde{v}_i(x) = \max\{v(T^i x), v(T^{i+1} x), \dots, v(T^k x)\},$$

and set $\tilde{v}_0 = \tilde{v}_1$. Then

$$\sum_{i=1}^k f(T^i x) v(T^i x) \leq \sum_{i=1}^k f(T^i x) \tilde{v}_i(x) \leq \sum_{i=0}^k f(T^i x) \tilde{v}_i(x)$$

and summation by parts shows that the term on the right equals

$$(5) \quad \left(\sum_{i=0}^k f(T^i x) \right) \tilde{v}_k(x) + \sum_{i=1}^k \left(\sum_{j=0}^{i-1} f(T^j x) \right) (\tilde{v}_{i-1}(x) - \tilde{v}_i(x)).$$

Now if $x \notin A$, $\sum_{j=0}^{i-1} f(T^j x) \leq \lambda \sum_{j=0}^{i-1} g(T^j x)$ for all $i \geq 1$, and (5) does not exceed

$$\lambda \left(\sum_{i=0}^k g(T^i x) \right) \tilde{v}_k(x) + \sum_{i=1}^k \left(\lambda \sum_{j=0}^{i-1} g(T^j x) \right) (\tilde{v}_{i-1}(x) - \tilde{v}_i(x)) = \lambda \sum_{i=0}^k g(T^i x) \tilde{v}_i(x)$$

again by partial summation. Now if $\{i_j\}_{j=1}^m$ denotes the points at which $\tilde{v}$ decreases, $i_0 = -1$, $i_{m+1} = k$, then $\tilde{v}_i(x) = v(T^{i_{j+1}}x)$ for $i_j + 1 \leq i \leq i_{j+1}$ so that

$$\begin{aligned} \sum_{i=0}^k g(T^i x) \tilde{v}_i(x) &= \sum_{j=0}^m \sum_{i=i_j+1}^{i_{j+1}} g(T^i x) \tilde{v}_i(x) \\ &= v(T^{i_1} x) \sum_{i=0}^{i_1} g(T^i x) + \sum_{j=1}^m v(T^{i_{j+1}} x) \sum_{i=i_j+1}^{i_{j+1}} g(T^i x) \\ &\leq v(T^{i_1} x) \sum_{i=0}^{i_1} g(T^i x) + \sum_{j=1}^m v(T^{i_{j+1}} x) \sum_{i=i_j}^{i_{j+1}} g(T^i x) \end{aligned}$$

and then since (2) shows

$$v(T^{i_1} x) \leq \frac{1}{2} \left(\frac{\sum_{i=0}^{i_1} \omega(T^i x)}{\sum_{i=0}^{i_1} g(T^i x)} \right) \quad \text{and} \quad v(T^{i_{j+1}} x) \leq \frac{1}{2} \left(\frac{\sum_{i=i_j+1}^{i_{j+1}} \omega(T^i x)}{\sum_{i=i_j}^{i_{j+1}} g(T^i x)} \right)$$

for $j \geq 1$, we have

$$\sum_{i=0}^k g(T^i x) \tilde{v}_i(x) \leq \frac{1}{2} \left(\sum_{i=0}^{i_1} \omega(T^i x) + \sum_{j=1}^m \sum_{i=i_j}^{i_{j+1}} \omega(T^i x) \right) \leq \sum_{j=0}^k \omega(T^j x).$$

This proves (4).

Now, following Derriennic [2], since T is conservative and ergodic,

$$\begin{aligned} \int_A f v d\mu &= \sum_{n=1}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^n T^{-i} A]} f(T^n x) v(T^n x) d\mu \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^k T^{-i} A] \cap T^{-(k+1)} A^c} f(T^n x) v(T^n x) d\mu \\ &= \sum_{k=1}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^k T^{-i} A] \cap T^{-(k+1)} A^c} \sum_{n=1}^k f(T^n x) v(T^n x) d\mu \end{aligned}$$

and by (4) this does not exceed

$$\begin{aligned} &\lambda \sum_{k=1}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^k T^{-i} A] \cap T^{-(k+1)} A^c} \sum_{n=0}^k \omega(T^n x) d\mu \\ &= \lambda \sum_{k=1}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^k T^{-i} A] \cap T^{-(k+1)} A^c} \left(\sum_{n=1}^k \omega(T^n x) + \omega(x) \right) d\mu \\ &= \lambda \left(\int_A \omega d\mu + \int_{A^c \cap T^{-1} A} \omega d\mu \right) \\ &= \lambda \int_{A \cup T^{-1} A} \omega d\mu. \end{aligned}$$

This completes the proof that (2) implies (1).

To prove that (1) implies (3), fix $n \geq 1$ and a constant $c > 1$. For each integer k and each rational $r \in (c^k, c^{k+1}]$ let

$$\Omega_{k,r} = \left\{ x : (g(T^n x))^{-1} > r \geq \left(\sum_{j=0}^n g(T^j x) \right)^{-1} > c^k \right\}.$$

Since $T^{-1}\Sigma = \Sigma$, for any $F \subset \Omega_{k,r}$ with $\mu(F) > 0$ there is $E_1 \in \Sigma$ with $F = T^{-n}E_1$ and since T is measure preserving, $\mu(E_1) = \mu(F)$. Then, since T is ergodic, there is $E \subset E_1$ with $\mu(E) > 0$ and $T^{-j}E$, $j = 0, \dots, n$ pairwise disjoint unless $\Omega = \bigcup_{j=0}^N T^{-j}I$ for an atom I with $T^{-j}I$, $j = 0, \dots, N$ pairwise disjoint for some $N < n$. (see [4, Lemma 2.2; 3, Corollaries 2.12 and 2.13]).

Leaving aside the atomic case for the moment, set $f = \chi_E$ and $\lambda = r$. Since $T^{-n}E \subset \Omega_{k,r}$ we have

$$S(\chi_E, g)(T^n x) \geq 1/g(T^n x) > r, \quad x \in T^{-n}E$$

so $T^{-n}E \subset T^{-n}A$, and since T is measure preserving,

$$(6) \quad \int_{T^{-n}E} v(T^n x) d\mu = \int_{T^{-n}A} \chi_E(T^n x) v(T^n x) d\mu = \int_A \chi_E v d\mu.$$

On the otherhand, if $x \notin \bigcup_{j=0}^n T^{-j}E$ and $j_1 < j_2 < \dots$ denotes those $j \geq 0$ for which $\chi_E(T^j x) = 1$, then

$$n+1 \leq j_1, \quad j_i + n+1 \leq j_{i+1} \quad \text{and} \quad T^{j_i} x \in E, \quad i = 1, 2, \dots$$

so

$$\sum_{j=1}^{j_m} g(T^j x) \geq \sum_{i=1}^m \sum_{j=j_i-n}^{j_i} g(T^j x) \geq \sum_{i=1}^m \frac{1}{r} = \frac{m}{r}$$

since $T^{j_i-n}x \in \Omega_{k,r}$. Thus we have

$$\frac{\sum_{j=0}^{j_m} \chi_E(T^j x)}{\sum_{j=0}^{j_m} g(T^j x)} \leq \frac{\sum_{j=1}^{j_m} \chi_E(T^j x)}{\sum_{j=1}^{j_m} g(T^j x)} \leq r$$

which shows $S(\chi_E, g)(T^i x) \leq r$ for $i = 0, 1$ and $x \notin \bigcup_{j=0}^n T^{-j}E$. Hence $A \cup T^{-1}A \subset \bigcup_{j=0}^n T^{-j}E$ so

$$(7) \quad \int_{A \cup T^{-1}A} \omega d\mu \leq \int_{\bigcup_{j=0}^n T^{-j}E} \omega d\mu = \int_{T^{-n}E} \left(\sum_{j=0}^n \omega(T^j x) \right) d\mu.$$

Thus (1) together with (6) and (7) yield

$$\int_{T^{-n}E} v(T^n x) d\mu \leq r \int_{T^{-n}E} \left(\sum_{j=0}^n \omega(T^j x) \right) d\mu.$$

Since F was arbitrary, this shows

$$v(T^n x) \leq r \sum_{j=0}^n \omega(T^j x)$$

for almost all $x \in \Omega_{k,r}$ and since $r \leq c^{k+1} \leq c(\sum_{j=0}^n g(T^j x))^{-1}$ for such x , we have

$$(8) \quad v(T^n x) \leq c \left(\frac{\sum_{j=0}^n \omega(T^j x)}{\sum_{j=0}^n g(T^j x)} \right)$$

for almost all $x \in \Omega_{k,r}$.

Suppose now that $\Omega = \bigcup_{j=0}^N T^{-j} I$ for an atom I with $T^{-j} I$, $j = 0, \dots, N$ pairwise disjoint. The argument which lead to (8) applies for $1 \leq n \leq N$. Since $T^{N+1} x = x$ for all $x \in \Omega$ in this case, if $n = m(N+1) + i$, $m \geq 1$, $0 \leq i \leq N$,

$$\sum_{j=0}^n \omega(T^j x) \geq m \sum_{j=i+1}^{N+1+i} \omega(T^j x) = m \sum_{j=0}^N \omega(T^{j+1+i} x)$$

and (8) for $n = N$ shows that this is greater than or equal to

$$\begin{aligned} c^{-1} v(T^{N+1+i} x) m \sum_{j=0}^N g(T^{j+1+i} x) &\geq (2c)^{-1} v(T^{N+1+i} x) \sum_{j=0}^n g(T^j x) \\ &= (2c)^{-1} v(T^n x) \sum_{j=0}^n g(T^j x). \end{aligned}$$

This shows that (8) holds for all $n \geq 1$ with c replaced by $2c$.

Since $c > 1$ was arbitrary and $\Omega = \bigcup_{k,r} \Omega_{k,r}$, this yields (3) and completes the proof of the Theorem.

The proof of the corollary follows exactly as in the unweighted case and, therefore, is omitted.

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