PROJECTIVITY OF THE WHITEHEAD PRODUCT IN SPHERES

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ABSTRACT. The Whitehead square $[l_{2^i-1}, l_{2^i-1}] \in \pi_{2^{i+1}-3}(S^{2^i-1})$ is shown to be projective for $i \leq 10$.

1. INTRODUCTION

A homotopy element in $\pi_n(X)$ is said to be projective if it can be represented by a map $f: S^n \rightarrow X$ that factors through the real projective space P^n , that is, the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ \sigma \searrow & \nearrow \bar{f} & \\ & P^n & \end{array}$$

commutes for some $\bar{f}$, where σ is the double covering map. The notion of projective homotopy elements was introduced by J. H. C. Whitehead [21]. Some research has been done on the projective homotopy theory and its applications in the last two decades. For this we refer to [17, §1].

Here we will be interested in the problem of the projectivity of the Whitehead products $[l_n, l_n] \in \pi_{2n-1}(S^n)$. The status of the problem so far is the following.

D. Randall proves in [18] that $[l_n, l_n]$ is not projective with n nor $n+1$ a power of two. Milgram and Zvengrowski show [16] that $[l_{2^m}, l_{2^m}]$, for $m \geq 1$, is projective iff $m = 1, 2, 3$. It remains to consider $[l_{2^i-1}, l_{2^i-1}]$. It is well known that $[l_{2^i-1}, l_{2^i-1}] = 0$ for $1 \leq i \leq 3$; so they are projective trivially. Adams Hopf invariant one theorem [1] implies $[l_{2^i-1}, l_{2^i-1}] \neq 0$ for $i \geq 4$. The problem of the projectivity of the family $\{[l_{2^i-1}, l_{2^i-1}]\}_{i \geq 4}$ is related to another well-known problem on the same family; namely, the so-called strong Kervaire invariant problem which asks whether $[l_{2^i-1}, l_{2^i-1}]$ ($i \geq 4$) can be halved. It is easy to see that if $[l_{2^i-1}, l_{2^i-1}]$ can be halved, then it is projective. Toda shows in [20] that $[l_{15}, l_{15}] = 2\sigma^2$, and Mahowald and Tangora prove in [15] that $[l_{31}, l_{31}] = 2\theta_4$; so $[l_{15}, l_{15}]$ and $[l_{31}, l_{31}]$ are projective. The conjecture for $i \geq 6$ is presently unknown.

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Our purpose here is to show that the answer to the projectivity problem about $[l_{2^i-1}, l_{2^i-1}]$ is positive in a few dimensions higher. The result is the following.

Theorem 1.1. *The Whitehead product $[l_{2^i-1}, l_{2^i-1}] \in \pi_{2^{i+1}-3}(S^{2^i-1})$ is projective for $i \leq 10$.*

We conjecture this result is true for all i .

2. PROOF

For $1 \leq k < l$ let P_k^l denote the stunted real projective space P^l/P^{k-1} . We will show the following.

Proposition 2.1. *Let $m = 2^i - 1$, $\bar{m} = 2^{i+1} - 2^{i-3} - 1$ ($i \geq 3$). Then for $i \leq 10$ there is a map $g: P_{\bar{m}}^{2m-1} \rightarrow S^m$ such that the composite*

$$S^{2m-1} \xrightarrow{\bar{\sigma}} P_{\bar{m}}^{2m-1} \xrightarrow{g} S^m$$

represents the Whitehead product $[l_m, l_m]$.

Here $\bar{\sigma}$ is the composite $S^{2m-1} \xrightarrow{\sigma} P^{2m-1} \xrightarrow{q} P_{\bar{m}}^{2m-1}$ with q the collapsing map. Note that the mapping cone of $\bar{\sigma}$ is $P_{\bar{m}}^{2m}$. Then, according to [4, Theorem 3.1], (2.1) is equivalent to

Proposition 2.1''. *Let $m, \bar{m}$ be as in (2.1). For $i \leq 10$, there is a stable map $\bar{g}: \sum P_{\bar{m}}^{2m} \rightarrow S^{m+1}$ such that in $X = S^{m+1} \cup_{\bar{g}} C \sum P_{\bar{m}}^{2m}$ the Steenrod operation*

$$Sq^{m+1} = Sq^{2^i}: H^{m+1}(X) = \mathbb{Z}_2 \rightarrow H^{2^{m+2}}(X) = \mathbb{Z}_2$$

is nonzero.

Here, and also for the rest of the paper, all cohomology or homology groups have $\mathbb{Z}_2$ coefficients.

Let $B(k)$ denote the k th 2-primary Brown-Gitler spectrum [7]. Recall that $H^*(B(k)) = A/A\{\chi(Sq^j) | j > k\}$ where A is the mod 2 Steenrod algebra and $\chi: A \rightarrow A$ is the canonical antiautomorphism. Then $\{\chi(Sq^j) | I = (i_1, \dots, i_k) \text{ admissible and } i_1 \leq k\}$ is a $\mathbb{Z}_2$ -base for $H^*(B(k))$. In particular $H^{2^{j+1}-1}(B(2^j)) = \mathbb{Z}_2$ is generated by $\chi(Sq^{2^j} Sq^{2^{j-1}} \cdots Sq^2 Sq^1)$ and $H^k(B(2^j)) = 0$ for $k > 2^{j+1} - 1$.

Consider the double loop space $\Omega^2 S^9$. There is a 2-primary stable decomposition

$$\Omega^2 S^9 \simeq \bigvee_{k=1}^{\infty} \sum^{7 \cdot k} B\left(\left[\frac{k}{2}\right]\right).$$

This is due to Brown-Peterson, Mahowald, and Snaith [8, 14, 19]. Let S^0 denote the sphere spectrum in stable dimension zero. Mahowald also shows the following in [14].

Proposition 2.2 [14]. *There is a stable map $f: \Omega^2 S^9 \rightarrow S^0$ such that if $f_i = f| \sum^{7 \cdot 2^{i-3}} B(2^{i-4})$ then in the mapping cone X_i of $F_i: \sum^{7 \cdot 2^{i-3}} B(2^{i-4}) \rightarrow S^0$ the Steenrod operation*

$$Sq^{2^i}: H^0(X_i) = \mathbb{Z}_2 \rightarrow H^{2^i}(X_i) = \mathbb{Z}_2$$

is nonzero for all $i \geq 3$ ($B(2^{i-4}) = B(0)$ if $i = 3$).

We will show the following.

Proposition 2.3. *Let $m = 2^i - 1$, $\bar{m} = 2^{i+1} - 2^{i-3} - 1$. For $3 \leq i \leq 10$ there is a stable map $\phi: \sum^{-\bar{m}} P_{\bar{m}}^{2m} \rightarrow B(2^{i-4})$ ($B(2^{i-4}) = B(0)$ if $i = 3$) such that $\phi^*: H^0(B(2^{i-4})) = \mathbb{Z}_2 \rightarrow H^0(\sum^{-\bar{m}} P_{\bar{m}}^{2m}) = \mathbb{Z}_2$ is nonzero.*

This implies $\phi^*: H^{2^{i-3}-1}(B(2^{i-4})) = \mathbb{Z}_2 \rightarrow H^{2^{i-3}-1}(\sum^{-\bar{m}} P_{\bar{m}}^{2m}) = \mathbb{Z}_2$ is nonzero since $\chi(Sq^{2^{i-4}} Sq^{2^{i-5}} \cdots Sq^2 Sq^1) = Sq^{2^{i-3}-1}: H^0(\sum^{-m} P_{\bar{m}}^{2m}) = \mathbb{Z}_2 \rightarrow H^{2^{i-3}-1}(\sum^{-\bar{m}} P_{\bar{m}}^{2m}) = \mathbb{Z}_2$ is nonzero [11]. Proposition 2.1', and therefore Theorem 1.1, follows from this and (2.2).

Note that the Spanier–Whitehead dual of

$$\sum^{-2^{i+1}+1} P_{\bar{m}}^{2m} = \sum^{-2^{i+1}+1} P_{2^{i+1}-2^{i-3}-1}^{2^{i+1}-2}$$

is

$$\sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+2^{i-3}}$$

[2, 3]. Then (2.3) is equivalent to

Proposition 2.3''. *Let*

$$X = \sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+2^{i-3}}.$$

For $3 \leq i \leq 10$ there is a stable map $\Phi: S^{2^{i-3}} \rightarrow B(2^{i-4}) \wedge X$ such that $\bar{\phi}^(1 \otimes \nu) \neq 0$ where ν is the top nonzero class of $H^*(X)$.*

To prove (2.3') we use the method in [6].

Let Y be a spectrum. Consider the mod 2 Adams spectral sequence $\{E_r^{*,*}(k)\}$ for computing $\pi_*^S(B(k) \wedge Y)$. An E_1 -term may be chosen as follows.

Let Λ be the lambda algebra of Kan et al. [5]. Λ is a bigraded algebra over $\mathbb{Z}_2$ generated by $\lambda_i \in \Lambda_i^1$ ($i \geq 0$) and has the set of admissible monomials $\{\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_s} | 2i_j \geq i_{j+1}\}$ as a $\mathbb{Z}_2$ -base. Let $\Lambda(k) \subset \Lambda$ be the $\mathbb{Z}_2$ -submodule generated by $\{\lambda_I | I = (i_1, \dots, i_s) \text{ admissible and } i_s \geq k\}$. Then

$$E_1^{s,t}(k) = \sum_p \Lambda_{t-s-p}^s(k) \otimes H_p(Y)$$

with differential

$$d_1(\lambda_I \otimes u) = \sum_{j \geq 0} \lambda_j \lambda_I \otimes u Sq^{j+1},$$

where the right action of the Steenrod algebra A on $H_*(Y)$ is obtained by transposing the left action of A on $H^*(Y)$.

Lemma 2.4 [6]. If $H_*(Y)$ is an unstable module over A , that is, $uSq^j = 0$ for $|u| < 2j$, then $d_1: E_1^{s,t}(k) \rightarrow E_1^{s+1,t}(k)$ is zero for $t - s \leq 2k + 1$.

Given an integer $n \geq 0$. Impose a condition " C_n " on Y as follows. For $n = 0$, Y satisfies " C_0 " if $H_l(Y) = 0$ for $l < 0$, and $H_*(Y)$ is an unstable A -module. For $n > 0$, Y satisfies " C_n " if there are a spectrum W that satisfies " C_{n-1} " and a map $f: Y \rightarrow W$ such that $f_*: H_*(Y) \rightarrow H_*(W)$ is monomorphic and for any $u \in H_l(Y)$ there exists $v \in H_{2l}(W)$ with $vSq^l = f_*(u)$. We also allow $n = \infty$; Y satisfies " C_∞ " if it satisfies " C_n " for all integers $n \geq 0$.

Examples.

(2.5.1) Y is said to have spacelike homology if it admits a map to the suspension spectrum of a CW complex, $f: Y \rightarrow W$ such that $f_*: H_*(Y) \rightarrow H_*(W)$ is a monomorphism. It is proved in [6] that if Y has spacelike homology then Y satisfies " C_∞ ". In particular, all suspension spectra of CW complexes satisfy " C_∞ ".

(2.5.2) For $i \geq 3$, the spectra

$$\sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+l} \quad \text{for } 1 \leq l \leq 2^{i-3}$$

satisfy " C_4 ". To see this we note that

$$H^* \left(\sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+l} \right), \quad \text{for } 1 \leq l \leq 2^{i+1},$$

is an unstable A -module, that

$$\sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+l} \subset \sum^{2^{i+1}} P_{-2^{i+1}+1}^0$$

and that, in $H^* \left(\sum^{2^{i+1}} P_{-2^{i+1}+1}^0 \right)$, $Sq^{8|u|} Sq^{4|u|} Sq^{2|u|} Sq^{|u|} u \neq 0$ for nonzero $u \in H^{|u|} \left(\sum^{2^{i+1}} P_{-2^{i+1}+1}^0 \right)$ with $1 \leq |u| \leq 2^{i-3}$.

Remark 2.6. Davis and Mahowald prove in [12] that

$$X = \sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+2^{i-3}}$$

and $P_1^{2^{i-3}}$ (for $i \geq 3$) have the same stable homotopy type if and only if $i \leq 6$. Thus, for $i \leq 6$, X has spacelike homology. It is not difficult to show that if X has spacelike homology then X has the stable homotopy type of $P_1^{2^{i-3}}$. Thus, for $i \geq 7$, X has no spacelike homology.

Proposition 2.7. For $0 \leq n \leq \infty$, if Y satisfies " C_n " then $d_r: E_r^{s,t}(k) \rightarrow E_r^{s+r,t+r-1}(k)$ is zero for $t - s \leq 2k + 1$ and $r \leq n + 1$.

The proof is given later.

Applying (2.7) to $Y = S^0$, we see, by (2.5.1), $E_\infty^{s,t}(k) = \Lambda_{t-s}^s(k)$ for $t - s \leq 2k$. Brown and Peterson prove in [9] the following sharper result.

Proposition 2.8 [9]. *As a group, $\pi_{t-s}^s(B(k)) \simeq \Lambda_{t-s}^s(k)$ for $t - s < 2k$.*

Now we deduce from (2.7) and (2.8) Proposition 2.3'.

Proof of Proposition 2.3'. We apply the spectral sequence to

$$Y = X = \sum^{2^{i+1}} P_{-2^{i+1}+1}^{-2^{i+1}+2^{i-3}}$$

in (2.3'). Let u be the nonzero class of $H_{2^{i-3}}(Y) = \mathbb{Z}_2$. Then $1 \otimes u \in E_1^{0, 2^{i-3}}(2^{i-4})$. To prove (2.3') is to show

$$(*) \quad 1 \otimes u \text{ persists to } E_\infty^{0, 2^{i-3}}(2^{i-4}) \quad \text{for } i \leq 10.$$

By (2.5.2) and (2.7), $d_r(1 \otimes u) = 0$ for $r \leq 5$. We have to show $d_r(1 \otimes u) = 0$ for $r \geq 6$.

Recall $\Lambda_{2^{i-3}-1-p}^r(2^{i-4})$ has $\{\lambda_I | I = (i_1, \dots, i_r) \text{ admissible, } i_r \geq 2^{i-4}, \text{ and } |I| = 2^{i-3} - 1 - p\}$ as a $\mathbb{Z}_2$ -base. One easily sees, then, that $\Lambda_{2^{i-3}-1-p}^r(2^{i-4}) = 0$ for $r \geq 6$, $3 \leq i \leq 9$, and $1 \leq p \leq 2^{i-3}$. Thus $d_r(1 \otimes u) = 0$ for $r \geq 6$ since

$$\begin{aligned} d_r(1 \otimes u) &\in E_r^{r, 2^{i-3}+r-1}(2^{i-4}) \subset E_1^{r, 2^{i-3}+r-1}(2^{i-4}) \\ &= \sum_p \Lambda_{2^{i-3}-1-p}^r(2^{i-4}) \otimes H_p(Y) = 0. \end{aligned}$$

This proves (*) for $i \leq 9$.

For $i = 10$, $r \geq 6$, $1 \leq p \leq 2^{i-3} = 2^7$, the $\mathbb{Z}_2$ -module $\Lambda_{2^{i-3}-1-p}^r(2^{i-4}) = \Lambda_{2^7-1-p}^r(2^6)$ is zero except for $r = 6$, $p = 1$ in which case $\Lambda_{2^7-2}^6(2^6) = \mathbb{Z}_2$ and is generated by $\lambda_2 \lambda_4 \lambda_8 \lambda_{16} \lambda_{32} \lambda_{64}$. So to prove (*) for $i = 10$ it suffices to show

$$d_6(1 \otimes u) = 0 \neq \lambda_2 \lambda_4 \lambda_8 \lambda_{16} \lambda_{32} \lambda_{64} \otimes u_1,$$

where $u_1 \neq 0 \in H_1(Y) = \mathbb{Z}_2$.

Consider

$$X' = \sum^{2^{11}} P_{-2^{11}+2}^{-2^{11}+2^7} = X/S^1 = \sum^{2^{11}} P_{-2^{11}+1}^{-2^{11}+2^7}/S^1.$$

Note that the Spanier-Whitehead dual of X' (resp. X) is

$$\sum^{-2^{11}+1} P_{2^{11}-2^7-1}^{2^{11}-3} \quad (\text{resp. } \sum^{-2^{11}+1} P_{2^{11}-7-1}^{2^{11}-2}).$$

Also note that X' satisfies " C_4 ". We apply (2.7) to X' . The analysis above shows that $1 \otimes u' \in E_1^{0, 2^7}(2^6)$ persists to E_∞ where $u' \neq 0 \in H_{2^7}(X') = \mathbb{Z}_2$. This implies there is a stable map

$$\sum^{-2^{11}+2^7+1} P_{2^{11}-2^7-1}^{2^{11}-3} \xrightarrow{h} B(2^6)$$

that is nonzero in $H_0(\)$. Let $\tau: S^{2^7-2} \rightarrow \sum^{-2^{11}+2^7+1} P_{2^{11}-2^7-1}^{2^{11}-3}$ be the attaching map for $\sum^{-2^{11}+2^7+1} P_{2^{11}-2^7-1}^{2^{11}-2}$. Then $d_6(1 \otimes u) = \lambda_2 \lambda_4 \lambda_8 \lambda_{16} \lambda_{32} \lambda_{64} \otimes u_1$ if and

only if the composite

$$S^{2^7-2} \xrightarrow{\tau} \sum^{-2^{11}+2^7+1} P_{2^{11}-2^7-1}^{2^{11}-3} \xrightarrow{h} B(2^6)$$

is detected by $\lambda_2\lambda_4\lambda_8\lambda_{16}\lambda_{32}\lambda_{64}$.

Consider the mapping cone $Z = (S^{2^7-2} \vee S^{2^7-1}) \cup e^{2^7}$ of $S^{2^7-1} \xrightarrow{(\eta, 2i)} S^{2^7-2} \vee S^{2^7-1}$. The map τ is the restriction of the attaching map

$$\bar{\tau}: Z \rightarrow \sum^{-2^{11}+2^7+1} P_{2^{11}-2^7-1}^{2^{11}-3} \quad \text{for } \sum^{-2^{11}+2^7+1} P_{2^{11}-2^7-1}^{2^{11}}.$$

If $h\tau$ is detected by $\lambda_2\lambda_4\lambda_8\lambda_{16}\lambda_{32}\lambda_{64}$ then $[h\bar{\tau}]_{S^{2^7-1}} \in \pi_{2^7-1}(B(2^6))$ is an element with $2[h\bar{\tau}]_{S^{2^7-1}}$ detected by $\lambda_1\lambda_2\lambda_4\lambda_8\lambda_{16}\lambda_{32}\lambda_{64} \neq 0$ since λ_1 detects η . In particular, $2[h\bar{\tau}]_{S^{2^7-1}} \neq 0$. But, by Proposition 2.8, all elements of $\pi_{2^7-1}(B(2^6))$ have order at most 2. Thus $h\tau$ cannot be detected by $\lambda_2\lambda_4\lambda_8\lambda_{16}\lambda_{32}\lambda_{64}$ that is, $d_6(1 \otimes u) = 0$. This proves (*) for $i = 10$. This completes the proof of Proposition (2.3'). Q.E.D.

The method of the proof of Proposition 2.7 is the one given in [6] where only spectra having space-like homology are considered. We recall the proof.

There are cofibrations [10, 14]

$$B(l-1) \wedge Y \xrightarrow{\beta} B(l) \wedge Y \xrightarrow{\alpha} \sum^l B\left(\left[\frac{l}{2}\right]\right) \wedge Y$$

natural in Y . β and α induce $\beta_*: E_r^{s,t}(l-1) \rightarrow E_r^{s,t}(l)$ and $\alpha_*: E_r^{s,t}(l) \rightarrow E_r^{s,t-l}([l/2])$. They have the following properties.

1. $\beta_*: E_1^{s,t}(l-1) \rightarrow E_1^{s,t}(l)$ is given by

$$\beta_*(\lambda_l \otimes u) = \lambda_l \otimes u.$$

2. (i) For $t-s < 2l$, $\alpha_*: E_1^{s,t}(l) \rightarrow E_1^{s,t-l}([l/2])$ is zero and any $\lambda_l \otimes u \in E_1^{s,t}(l)$ pulls back uniquely to $E_1^{s,t}(l-1)$ (as described in 1).

(ii) If $|\lambda_{2l} \otimes u| = 2l$ and $u \in H_{2p}(Y)$ then

$$\alpha_*(\lambda_{2l} \otimes u) = \lambda_l \otimes u Sq^p.$$

Proposition 2.7 is proved by induction on n and k . The result is obvious for $k = 0$. It is also true for $n = 0$ by Lemma 2.4. Suppose $n > 0$, $k > 0$, and suppose the result is true for either $k' < k$ or $n' < n$.

Let $\lambda_l \otimes u \in E_1^{s,t}(k)$ with $q = |\lambda_l \otimes u| = t-s \leq 2k+1$. If $q < 2k$ then, by 1, 2(i), $\lambda_l \otimes u$ lies in $E_1^{s,t}(k-1)$. Since $q \leq 2(k-1)+1$, by inductive hypothesis and the fact that β_* commutes with all d_r , we see $d_r(\lambda_l \otimes u) = 0$ for $r \leq n+1$. Thus we may assume $q = 2k$ or $2k+1$.

By assumption, there exist a spectrum W that satisfies " C_{n-1} ", a map $f: Y \rightarrow W$ monomorphic in homology, and a class $\nu \in H_{2|u|}(W)$ with $\nu Sq^{|u|} = f_*(u)$. Let $\{\bar{E}_r^{*,*}(m)\}$ be the Adams spectral sequence for $\pi_*^s(B(m) \wedge W)$.

Now both Y and W satisfy " C_{n-1} ". So, by inductive hypothesis, $E_1^{s,t}(k) = E_{n+1}^{s,t}(k)$, $\overline{E}_1^{s,t}(k) = \overline{E}_{n+1}^{s,t}(k)$ for $t-s \leq 2k$, $\overline{E}_1^{s,t}(q) = \overline{E}_{n+1}^{s,t}(q)$ ($q = |\lambda_I \otimes u| = 2k$ or $2k+1$) for $t-s \leq 2q$ and $E_{n+1}^{s,t}(k)$ (resp. $\overline{E}_{n+1}^{s,t}(k)$) is a quotient $\mathbb{Z}_2$ -module of $E_1^{s,t}(k)$ (resp. $\overline{E}_1^{s,t}(k)$) if $t-s = 2k+1$. Thus the induced map $f_*: E_{n+1}^{s,t}(k) \rightarrow \overline{E}_{n+1}^{s,t}(k)$ is also monomorphic for $t-s \leq 2k$. Let $\pi: E_1^{s,t}(k) \rightarrow E_{n+1}^{s,t}(k)$ (resp. $\overline{\pi}: \overline{E}_1^{s,t}(k) \rightarrow \overline{E}_{n+1}^{s,t}(k)$) be the quotient map if $t-s = 2k+1$ and the identity map if $t-s \leq 2k$. We have

$$\begin{aligned} (f_* \circ d_{n+1} \circ \pi)(\lambda_I \otimes u) &= (\overline{d}_{n+1} \circ \overline{\pi})(\lambda_I \otimes f_*(u)) \\ &= (\overline{d}_{n+1} \circ \overline{\pi})(\lambda_I \otimes \nu Sq^{|u|}) \\ &= (\overline{d}_{n+1} \circ \overline{\pi} \circ \alpha_*)(\lambda_{2I} \otimes \nu) \quad (\text{by 2(ii)}) \\ &= (\alpha_* \circ \overline{d}_{n+1} \circ \overline{\pi})(\lambda_{2I} \otimes \nu) \\ &= 0. \end{aligned}$$

Here the last step follows by 2(i) since

$$\overline{d}_{n+1}(\overline{\pi}(\lambda_{2I} \otimes \nu)) \in \overline{E}_{n+1}^{s+n+1, s+n+2q}(q) = \overline{E}_1^{s+n+1, s+n+2q}(q)$$

and

$$\alpha_*(\overline{d}_{n+1}(\overline{\pi}(\lambda_{2I} \otimes \nu))) \in \overline{E}_{n+1}^{s+n+1, s+q+n}(k) = \overline{E}_1^{s+n+1, s+q+n}(k).$$

Since $d_{n+1}(\pi(\lambda_I \otimes u))$ lies in $\overline{E}_{n+1}^{s,t}(k)$ with $t-s = q-1 \leq 2k$ and since $f_*: E_{n+1}^{s,t}(k) \rightarrow \overline{E}_{n+1}^{s,t}(k)$ is monomorphic for $t-s \leq 2k$ we see that

$$d_{n+1}(\pi(\lambda_I \otimes u)) = 0.$$

Thus $d_{n+1}(E_{n+1}^{s, s+q}) = 0$ since π is onto. This completes the inductive proof of Proposition 2.7.

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