A FINITENESS PROPERTY OF AFFINE ALGEBRAS

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ABSTRACT. Let A be a finitely presented reduced commutative k -algebra and J an ideal in A . If A embeds in A/J then $J = 0$.

1. STATEMENT OF THE PROBLEM

We consider the following situation that arose in a technical context in [B].

K = a commutative ring.

A = a commutative K -algebra.

J = an ideal in A .

$f: A \rightarrow A/J$ is an injective K -algebra homomorphism.

We want to conclude that $J = 0$ under suitable finiteness assumptions.

(1.1) **Theorem.** *Assume:*

(FP) A is a finitely presented K -algebra; and

(Red) A is reduced (i.e., all nilpotent elements are zero).

Then $J = 0$.

The following examples illustrate the necessity of the hypotheses (FP) and (Red).

(1.2) **Example.** Let $(x_n)_{n \geq 0}$ be a sequence of indeterminates, $A = K[(x_n)_{n \geq 0}]$, $J = Ax_0$, so that $A/J = K[x_1, x_2, x_3, \dots]$, and define $f: A \rightarrow A/J$ by $f(x_n) = x_{n+1} \pmod{J}$ for $n \geq 0$. Here A is reduced (and finitely related), but not finitely generated. I do not know if one can relax (FP) and require only that A be finitely generated.

(1.3) **Example.** Let $R = K[x]$, a polynomial ring in one variable. Let $M = Ru \oplus Rv$, a free R -module with basis u, v . Put $A = R \oplus M$ with $M^2 = 0$, and let $J = Av = Rv$. Then $A/J = R \oplus Ru$ with $u^2 = 0$. Define $f: A \rightarrow A/J$ to be the K -algebra homomorphism such that $f(x) = x^2$, $f(u) = u$, and $f(v) = xu$ then

$$f(A) = K[x^2] + K[x^2]u + K[x^2]xu.$$

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Since $K[x] = K[x^2] \oplus K[x^2]x$, the above decomposition of $f(A)$ is a direct sum, hence f is injective. Here $A = K[x, u, v]/(u^2, uv, v^2)$ is a finitely presented K -algebra, but A is not reduced.

The proof of (1.1) is carried out first in the case when K is noetherian, and then concluded by a reduction to the noetherian case.

2. PROOF OF (1.1) WHEN K IS NOETHERIAN

We may assume that $K \subset A$, so K is also reduced. To show that $J = 0$ we can localize, and so assume that K is local, say with maximal ideal m , and further that $J_p = 0$ for all primes $p \neq m$ in K . It follows then that J is annihilated by some power of m , and hence by m itself, since A is reduced:

$$(1) \quad mJ = 0.$$

Next we may assume that J is maximal among ideals of A for which there is a K -algebra embedding $f: A \rightarrow A/J$. Since $f(A) \cong A$ is reduced the composite $A \xrightarrow{f} A/J \rightarrow A/\sqrt{J}$ is injective, so $J = \sqrt{J}$, by maximality. Write

$$(2) \quad J = Q_1 \cap \cdots \cap Q_r$$

as an irredundant intersection of prime ideals. Put

$$P_i = f^{-1}(Q_i) \in \text{Spec}(A).$$

The injectivity of f translates into the condition:

$$(3) \quad P_1 \cap \cdots \cap P_r = 0.$$

It follows that all minimal primes of A occur among the P_i . Reindex so that each minimal prime of A occurs exactly once among the $P_1, \dots, P_s$ (some $s \leq r$). Putting $J' = Q_1 \cap \cdots \cap Q_s \supset J$ we then see that the composite $A \xrightarrow{f} A/J \rightarrow A/J'$ is injective, so again by maximality, $J = J'$. Thus,

$$(4) \quad \begin{cases} \text{each minimal prime of } A \text{ occurs} \\ \text{exactly once among } P_1, \dots, P_r. \end{cases}$$

Now f induces a K -algebra embedding

$$f_i: A/P_i \hookrightarrow A/Q_i$$

and it follows that

$$(5) \quad P_i \text{ and } Q_i \text{ lie over the same prime } p_i \text{ of } K.$$

For $i \in E = \{1, \dots, r\}$ we can, in view of (4), choose $\sigma(i) \in E$ so that $P_{\sigma(i)} \subset Q_i$, and hence also, by (5),

$$(6) \quad p_{\sigma(i)} \subset p_i.$$

Let $E_m = \{i \in E | p_i = m\}$ and $E'_m = E - E_m$. Recall that m is the maximal ideal of K and we have

$$(1') \quad 0 = mJ = m \cdot (Q_1 \cap \cdots \cap Q_r).$$

It follows from (1') that Q_i ($i \in E'_m$) are the minimal primes of A not containing m . Consequently

$$(7) \quad Q_i = P_{\sigma(i)} \quad \text{for } i \in E'_m$$

and, therefore, since (2) is irredundant

$$(8) \quad \sigma(E_m) \subset E_m.$$

Case 1. $E_m = E$. Then $m = \bigcap_i p_i \subset \bigcap_i P_i = 0$, so K is a field. For a finitely generated K -algebra B put $t(B) = \sum_p \text{transc deg}_K(B/P)$, where P ranges over the minimal primes of B . Then we have $t(A/J) \leq t(A) \leq t(A/J)$, the latter because of the embedding of A into A/J . Hence $t(A/J) = t(A)$. Since A is reduced it follows easily that $J = 0$, whence our result in this case.

Case 2. $E_m \neq E$. It follows from (8) that

$$L = \bigcap_{i \in E_m} P_i \subset J' = \bigcap_{i \in E_m} Q_i.$$

Hence f induces a K -algebra embedding

$$A/L \rightarrow A/J' = (A/L)/(J'/L).$$

By induction on $r = \text{Card}(E)$ we conclude that $J'/L = 0$, that is $J' = L$ and therefore, by irredundancy,

$$(9) \quad Q_i = P_{\sigma(i)} \quad \text{for all } i \in E_m.$$

Combined with (7), this gives $Q_i = P_{\sigma(i)}$ for all $i \in E$; hence $\sigma: E \rightarrow E$ is a permutation, and $J = \bigcap_i Q_i = \bigcap_i P_i = 0$.

In applying induction on r , we need to check the initial case, $r = 1$ when $J = Q_1 \supset P_1 = 0$. Since $mJ = 0$ either $m \neq 0$, so $J = 0$ (A is an integral domain) or $m = 0$, and we are as in Case 1 above.

3. REDUCTION TO THE NOETHERIAN CASE

Write $A = S/L$ where $S = K[x_1, \dots, x_n]$ is a polynomial K -algebra, and $L = \sum_{i=1}^m Sg_i$ is a finitely generated ideal. We have $J = J_1/L$ for some ideal J_1 of S . The K -algebra embedding

$$f: A = S/L \rightarrow A/J = S/J_1$$

sends $(x_i \bmod L)$ to $(p_i \bmod J_1)$ for some $p_i \in S$ ($i = 1, \dots, n$). We first treat:

Case 1. J_1 is a finitely generated ideal.

Suppose $J_1 = \sum_{i=1}^r Sh_i$. Let K' be a "large" finitely generated (hence noetherian) subring of K so that $S' = K'[x_1, \dots, x_n]$ contains all g_i, h_i, p_i . Put $L' = \sum_{i=1}^m S'g_i$ and $J'_1 = (\sum_{i=1}^r S'h_i) + L'$. Since f is well defined we have

$$g_i(p_1, \dots, p_n) \in J_1 \quad (i = 1, \dots, m).$$

After enlarging K' , if necessary, we obtain

$$g_i(p_1, \dots, p_n) \in J'_1 \quad (i = 1, \dots, m).$$

Thus we have a well-defined K' -algebra homomorphism

$$f': A' = S'/L' \rightarrow S'/J'_1 = A'/(J'_1/L').$$

Let $\text{Ker}(f') = H/L'$, where the ideal H of S' is finitely generated, since S' is noetherian; suppose $H = \sum_{i=1}^t S' u_i$. Since $f = K \otimes_{K'} f'$ is injective we have $u_i \in L$ ($i = 1, \dots, t$). By enlarging K' again we can make $u_i \in L'$ ($i = 1, \dots, t$), and render f' injective.

Now by the noetherian case of (1.1) we conclude that $J'_1/L' = 0$, that is $J'_1 = L'$. Since $J_1 = S \cdot J'_1$ we conclude also that $J_1 = L$, hence $J = 0$, as we set out to show.

To finish the proof we now make a *reduction to Case 1*. Let $f(A) = S_1/J_1$, where S_1 is a sub- K -algebra of S . Since A is finitely generated we can choose a finitely generated K -algebra $S'_1 \subset S_1$ that projects onto S_1/J_1 . Putting $J'_1 = J_1 \cap S'_1$ we then have $A \cong f(A) = S'_1/J'_1$. Since A is finitely presented and S'_1 is finitely generated (as K -algebras) it follows that J'_1 is a finitely generated ideal of S'_1 . Thus $(J'_1 \cdot S) + L$ is a finitely generated S -ideal. Let J_2 be any finitely generated S -ideal such that $(J'_1 \cdot S) + L \subset J_2 \subset J_1$. Since $J_1 \cap S'_1 = J'_1$ it follows that $J_2 \cap S'_1 = J'_1$ also. Hence the image of S'_1 in $S/J_2 = A/(J_2/L)$ is isomorphic to $S'_1/J'_1 \cong A$. Thus we can apply Case 1 to A with $J = J_1/L$ replaced by J_2/L , and conclude that $J_2 = L$. However, we are free to enlarge J_2 by adding any new generator from J_1 . Thus we see that $J_1 = L$, that is $J = 0$, as we set out to confirm.

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