A UNICELLULAR UNIVERSAL QUASINILPOTENT WEIGHTED SHIFT

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ABSTRACT. For a suitably chosen sequence of weights $\{\alpha_n\}$, the unilateral weighted shift Q on l^p ($1 \leq p < \infty$), defined by $Qe_n = \alpha_n e_{n+1}$ ($n \geq 1$), is a unicellular quasinilpotent operator such that Q^k is not compact for any power $k \geq 1$. As a corollary, several applications to approximation of Hilbert space operators are given.

Inductively, define $\gamma_1 = 1$, $\gamma_2 = \frac{1}{4}$, $\gamma_n = (\gamma_1 \gamma_2 \cdots \gamma_{n-1})^n$, and let $\{\alpha_n\}$ be the sequence

$\gamma_1, \gamma_2, \dots, \gamma_9, \gamma_1, \gamma_2, \dots, \gamma_{90}, \gamma_1, \gamma_2, \dots, \gamma_{900}, \gamma_1, \gamma_2, \dots, \gamma_{9,000}, \gamma_1, \dots$

Let Q be the unilateral weighted shift defined by $Qe_n = \alpha_n e_{n+1}$ ($n \geq 1$) with respect to the canonical basis $\{e_n\}_{n \geq 1}$ of l^p ($1 \leq p < \infty$), and let Q_r be similarly defined, with the first 10^r weights replaced by zeros. If $10^{r-1} \leq k < 10^r$, then a straightforward calculation shows that

$$\|(Q_r)^k\| = \text{essential norm}(Q_r)^k = \prod_{j=1}^k \gamma_j > 0$$

(so that $(Q_r)^k$ is not a compact operator), and therefore the spectral radius of Q_r cannot exceed

$$\left(\prod_{j=1}^k \gamma_j \right)^{1/k} < (\gamma_k)^{1/k} \rightarrow 0 \quad (k \rightarrow \infty).$$

Since $Q - Q_r$ has finite rank and the spectrum of a unilateral weighted shift is always connected, we conclude that

Q is quasinilpotent and Q^k is not compact for any $k \geq 1$.

(The reader is referred to the monograph [13] for general information about unilateral weighted shifts.)

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It only remains to show that Q is unicellular, that is, the invariant subspace lattice of Q is the chain

$$\{0\}, \mathcal{M}_t = \bigvee \{e_n\}_{n \geq t} \quad (t \geq 1; \mathcal{M}_1 = l^p).$$

If $1 \leq m \leq [n/2]$ and $n - m + 1 \leq 10^r < n$, then $n - m + s = 10^r$ for some s ($1 \leq s \leq m - 1$) and

$$\begin{aligned} \prod_{j=1}^m \frac{\alpha_{n-j+1}}{\alpha_j} &= \left(\frac{\alpha_{n-m+s+1} \alpha_{n-m+s+2} \cdots \alpha_n}{\alpha_1 \alpha_2 \cdots \alpha_{m-s}} \right) \left(\frac{\alpha_{n-m+1} \alpha_{n-m+2} \cdots \alpha_{n-m+s}}{\alpha_{m-s+1} \alpha_{m-s+2} \cdots \alpha_m} \right) \\ &\leq \frac{\alpha_{n-m+1} \alpha_{n-m+2} \cdots \alpha_{n-m+s}}{\alpha_{m-s+1} \alpha_{m-s+2} \cdots \alpha_m} \leq (\gamma_{10^r - 10^{r-1}})^{1-1/(10^r - 10^{r-1})} \leq 4^{-m} \end{aligned}$$

because $m < n - m + s$.

If $1 \leq m \leq [n/2]$ and $10^r < n - m + 1 < n \leq 10^{r+1}$, then

$$\prod_{j=1}^m \frac{\alpha_{n-j+1}}{\alpha_j} \leq \begin{cases} 1, & \text{if } m \leq 10, \\ 4^{10-m}, & \text{if } m > 10. \end{cases}$$

Thus, we have

$$\begin{aligned} \sup_n \sum_{m=0}^n \prod_{j=1}^m \frac{\alpha_{n-j+1}}{\alpha_j} &\leq 2 + 2 \sup_n \sum_{m=1}^{[n/2]} \prod_{j=1}^m \frac{\alpha_{n-j+1}}{\alpha_j} \\ &< 2 + 2 \cdot 4^{10} \sum_{m=1}^{\infty} 4^{-m} < 4^{10} < \infty. \end{aligned}$$

It follows from [4, 11] (or [12]) that Q is a strictly cyclic operator; that is, $l^p = \mathcal{A}(Q)e_1$, where $\mathcal{A}(Q)$ is the uniform closure of the polynomials in Q and I . Thus, $\mathcal{A}(Q)$ is a maximal abelian Banach algebra of operators (acting on l^p), whose Gelfand spectrum coincides with $\sigma(Q) = \{0\}$ [7, Theorem 4]. It readily follows that $\mathcal{A}(Q)$ has exactly one maximal ideal $\mathcal{A}_1(Q)$. Under the isomorphism of Banach spaces defined by $A \rightarrow Ae_1$ (from $\mathcal{A}(Q)$ onto l^p), (closed) ideals of $\mathcal{A}(Q)$ correspond to invariant subspaces of Q ; in particular, $\mathcal{A}_1(Q)e_1 = \mathcal{M}_2$ includes all the nontrivial invariant subspaces of Q .

Observe that $Q|_{\mathcal{M}_t}$ is a quasinilpotent unilateral weighted shift with weight sequence $\{\alpha_n^{(t)} = \alpha_{n+t}\}_{n \geq 1}$. A cumbersome calculation (similar to the above one) shows that

$$\sup_n \sum_{m=0}^n \prod_{j=1}^m \frac{\alpha_{n-j+1}^{(t)}}{\alpha_j^{(t)}} \leq C_t < \infty,$$

hence we conclude that every nontrivial invariant subspace of $Q|_{\mathcal{M}_t}$ is a subspace of $\mathcal{M}_{t+1}$ ($t \geq 1$).

It readily follows that

$$\text{Lat } Q = \{\{0\}; \mathcal{M}_t, t \geq 1\}.$$

The existence of these kinds of operators affirmatively answers a question of J. B. Conway (personal communication).

1. The above argument (if $Q|_{\mathcal{M}_t}$ is strictly cyclic for all $t \geq 1$, then Q is unicellular) is due to R. Gellar [5].

2. For $1 < p < \infty$, the adjoint of Q is the backward weighted shift R , defined by $Re_1 = 0$, $Re_n = \alpha_{n-1}e_{n-1}$ ($n > 1$; R acts on l^q , where $\frac{1}{p} + \frac{1}{q} = 1$). Clearly,

$$\text{Lat } R = \left\{ \{0\}; \bigvee \{e_n\}_{n=1}^t, t \geq 1; l^q \right\}.$$

3. In the Hilbert space case ($p = 2$), Q and R are “universal quasiniptents”; that is, the closure of the similarity orbit of Q (or R) coincides with the set of limits of nilpotent operators; more precisely,

$$\mathcal{S}(Q)^- = \mathcal{S}(R)^- = \{A \in \mathcal{L}(l^2): \text{the spectrum } \sigma(A) \text{ and the essential spectrum } \sigma_e(A), \text{ of } A, \text{ are connected sets containing the origin, and } \text{ind}(\lambda - A) = 0 \text{ for each complex } \lambda \text{ such that } \lambda - A \text{ is a semi-Fredholm operator}\}$$

(see [1, 2, 5, 6] or [7, Chapter 8]). (Here $\mathcal{S}(T) := \{WTW^{-1}: W \text{ is invertible}\}$ denotes the similarity orbit of the operator T , and the upper bar indicates norm-closure.)

By using this last observation, several known results in the literature can be given very simple proofs. Two examples:

Corollary 1 (D. W. Hadwin [6]). *The norm-closure of the set of all unicellular operators on a Hilbert space $\mathcal{H}$ coincides with*

$$\{A \in \mathcal{L}(\mathcal{H}): \sigma(A) \text{ and } \sigma_e(A) \text{ are connected sets and } \text{ind}(\lambda - A) = 0 \text{ for all } \lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is semi-Fredholm}\}.$$

Proof. Let $\mathcal{W}$ denote the above described set. The continuity properties of the Riesz–Dunford functional calculus and the stability properties of the semi-Fredholm operators indicate that every unicellular operator on $\mathcal{H}$ is a member of $\mathcal{W}$ (see, e.g., [10, Chapter 1]; this inclusion is valid in every Banach space). Since $\mathcal{W}$ is a closed subset of $\mathcal{L}(\mathcal{H})$, it must include the closure of the unicellular operators.

On the other hand, if $A \in \mathcal{W}$ and $\mu \in \sigma_e(A)$, then $A - \mu \in \mathcal{S}(Q)^-$. Thus,

$$\mathcal{W} = \bigcup_{\mu \in \mathbb{C}} \{\mu + T: T \in \mathcal{S}(Q)^-\}.$$

Since Q is unicellular, we are done. $\square$

A subspace $\mathcal{M}$ is *stably invariant* for $T \in \mathcal{L}(\mathcal{H})$ if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|T - T'\| < \delta$, then T' has an invariant subspace $\mathcal{M}'$ satisfying

$$\|P_{\mathcal{M}} - P_{\mathcal{M}'}\| < \varepsilon,$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{M}$.

Corollary 2 (C. Apostol, C. Foiaş, and N. Salinas [3]; also see [2, Appendix II]). *Assume that $\sigma(T)$ and $\sigma_e(T)$ are connected sets and $\text{ind}(\lambda - T) = 0$ for*

each $\lambda \in \mathbb{C}$ such that $\lambda - T$ is a semi-Fredholm operator. Then T is “stably transitive”; that is, the only stably invariant subspaces of T are the trivial ones ($\{0\}$ and $\mathcal{H}$).

Proof. Let $\mu \in \sigma_e(T)$; then

$$T - \mu \in \mathcal{S}(Q)^- = \mathcal{S}(R)^-.$$

Thus, for each $\delta > 0$ there exist Q' similar to Q and R' similar to R such that

$$\|T - Q'\| < \delta \quad \text{and} \quad \|T - R'\| < \delta.$$

Let $\mathcal{Q}$ and $\mathcal{R}$ be any two nontrivial subspaces of $\mathcal{H}$ invariant under Q' and, respectively, under R' . Since $\mathcal{Q}$ is necessarily finite-dimensional and $\mathcal{R}$ is necessarily of finite codimension, we have $\|P_{\mathcal{Q}} - P_{\mathcal{R}}\| \geq 1$. Furthermore,

$$\max[\|P_{\mathcal{Q}} - P_{\mathcal{M}}\|, \|P_{\mathcal{R}} - P_{\mathcal{M}}\|] \geq 1$$

for each subspace $\mathcal{M}$ of $\mathcal{H}$!

Therefore, T cannot have any nontrivial stably invariant subspace. $\square$

It is clear from this last example that the operators Q and R can play a very interesting role in the still open problem of characterizing all the stably invariant subspaces of a given operator T .

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