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THE NORMAL HOLONOMY GROUP

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ABSTRACT. We prove that the restricted normal holonomy group of a submanifold of a space of constant curvature is compact and that the nontrivial part of its representation on the normal space is the isotropy representation of a semisimple Riemannian symmetric space.

1. INTRODUCTION

The restricted holonomy group of a Riemannian manifold is a compact Lie group, and its representation on the tangent space is a product of irreducible representations and a trivial one. This product is unique up to order (see, e.g., [K-N, §5]). Each one of the nontrivial factors is either an orthogonal representation of a connected compact Lie group which acts transitively on the unit sphere or it is the isotropy representation of a simple Riemannian symmetric space of rank ≥ 2 (see [B, S]).

We prove that, surprisingly, all these properties are also true for the representation on the normal space of the restricted normal holonomy group of any submanifold of a space of constant curvature. Moreover, the nontrivial part of this representation is the isotropy representation of a semisimple Riemannian symmetric space.

In order to prove this fact we define a tensor

$$\mathscr{R}^{\perp}: C^{\infty}(N, (M))^{3} \to C^{\infty}(N(M)),$$

which provides the same geometric information as the normal curvature tensor $R^{\perp}$ and has the algebraic properties of a Riemannian curvature tensor. The methods used here are then a slight modification of those of Simons in [S].

2. NORMAL CURVATURE

Let $(M^n, \langle\ ,\ \rangle)$ be a Riemannian connected manifold and let $i: M^n \to Q^N$ be an isometric immersion, where Q^N is of constant curvature.

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Let $N(M) \xrightarrow{\pi} M$ be the normal bundle over M induced by i. For the sake of simplicity the Riemannian metric on Q^N, as well as the usual metric on the fibers of $N(M)$, will also be denoted by $\langle\ ,\ \rangle$. By $C^\infty(N(M))$ we denote the C^∞ sections from M into $N(M)$.

Define the tensor

$$\mathscr{R}^\perp : C^\infty(N(M))^3 \to C^\infty(N(M))$$

by putting

$$\mathscr{R}_p^\perp(\xi_1, \xi_2)\xi_3 = \sum_{j=1}^n R_p^\perp(A_{\xi_1}(e_j), A_{\xi_2}(e_j))\xi_3,$$

$p \in M$, $\xi_1, \xi_2, \xi_3 \in N(M)_p$; where A is the shape operator, $R^\perp$ is the curvature operator of the normal connection $\nabla^\perp$ and $\{e_1, \dots, e_n\}$ is an aribitrary orthonormal basis of T_pM.

The above tensor was just defined in [O-S].

Given an Euclidean space $\mathbf{V}$ we will denote by $\mathscr{A}(\mathbf{V})$ the vector space of skew-symmetric endomorphisms of $\mathbf{V}$, with the usual inner product $(\ ,\ )$, i.e., $(A, B) = -\operatorname{trace}(A \circ B)$.

Lemma 2.1. *Assume the hypothesis and notation of this section. Then, for all* $\xi_1, \xi_2, \xi_3, \xi_4 \in C^\infty(N(M))$, *the following are verified*:

(i) $\mathscr{R}^\perp(\xi_1, \xi_2) = -\mathscr{R}^\perp(\xi_2, \xi_1)$,

(ii) $\mathscr{R}^\perp(\xi_1, \xi_2)\xi_3 + \mathscr{R}^\perp(\xi_2, \xi_3)\xi_1 + \mathscr{R}^\perp(\xi_3, \xi_1)\xi_2 = 0$,

(iii) $\langle\mathscr{R}^\perp(\xi_1, \xi_2)\xi_3, \xi_4\rangle = -\langle\xi_3, \mathscr{R}^\perp(\xi_1, \xi_2)\xi_4\rangle$,

(iv) $\langle\mathscr{R}^\perp(\xi_1, \xi_2)\xi_3, \xi_4\rangle = \langle\mathscr{R}^\perp(\xi_3, \xi_4)\xi_1, \xi_2\rangle = -\frac{1}{2}([A_{\xi_1}, A_{\xi_2}], [A_{\xi_3}, A_{\xi_4}])$.

Proof. The proof was given in [O-S], but we reproduce it completely. Let $p \in M$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM.

$$\begin{aligned}\langle\mathscr{R}_p^\perp(\xi_1, \xi_2)\xi_3, \xi_4\rangle &= \left\langle \sum_{j=1}^n R_p^\perp(A_{\xi_1}(e_j), A_{\xi_2}(e_j))\xi_3, \xi_4 \right\rangle \\ &= \sum_{j=1}^n \langle [A_{\xi_3}, A_{\xi_4}](A_{\xi_1}(e_j)), A_{\xi_2}(e_j)\rangle\end{aligned}$$

by the well-known formula, when the ambient space is of constant curvature,

$$\begin{aligned}&= \sum_{j=1}^n \langle A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}] \circ A_{\xi_1}(e_j), e_j\rangle \\ &= \operatorname{trace}_p(A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}] \circ A_{\xi_1}) \\ &= \tfrac{1}{2}\operatorname{trace}_p(A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}] \circ A_{\xi_1}) \\ &\quad + \tfrac{1}{2}\operatorname{trace}_p((A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}] \circ A_{\xi_1})^t)\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\operatorname{trace}_p(A_{\xi_1} \circ A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}]) \\
&\quad - \tfrac{1}{2}\operatorname{trace}_p(A_{\xi_1} \circ [A_{\xi_3}, A_{\xi_4}] \circ A_{\xi_2}) \\
&= \tfrac{1}{2}\operatorname{trace}_p\{A_{\xi_1} \circ A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}] - A_{\xi_2} \circ A_{\xi_1} \circ [A_{\xi_3}, A_{\xi_3}]\} \\
&= \tfrac{1}{2}\operatorname{trace}_p([A_{\xi_1}, A_{\xi_2}] \circ [A_{\xi_3}, A_{\xi_4}])
\end{aligned}
$$

which proves (i), (iii), and (iv).

As we have seen above,

$$
\begin{aligned}
\langle \mathscr{R}_p^{\perp}(\xi_1, \xi_2)\xi_3, \xi_4\rangle &= \operatorname{trace}_p(A_{\xi_2} \circ [A_{\xi_3}, A_{\xi_4}] \circ A_{\xi_1}) \\
&= \operatorname{trace}_p(A_{\xi_2} \circ A_{\xi_3} \circ A_{\xi_4} \circ A_{\xi_1}) \\
&\quad - \operatorname{trace}_p(A_{\xi_2} \circ A_{\xi_4} \circ A_{\xi_3} \circ A_{\xi_1}) \\
&= \operatorname{trace}_p(A_{\xi_1} \circ A_{\xi_2} \circ A_{\xi_3} \circ A_{\xi_4}) \\
&\quad - \operatorname{trace}_p(A_{\xi_3} \circ A_{\xi_1} \circ A_{\xi_2} \circ A_{\xi_4}).
\end{aligned}
$$

Summing over all cyclic permutations of $\{1, 2, 3\}$ we clearly obtain (ii). □

Observe that, from (iv), we have that $R_p^{\perp} = 0 \Leftrightarrow \mathscr{R}_p^{\perp} = 0$.

We have much more than this. The following result tells us that $\mathscr{R}^{\perp}$ carries the same geometrical information as $R^{\perp}$.

Proposition 2.2. *Assume the notation and assumptions of this section. Then, for all $p \in M$, the linear space of skew-symmetric endomorphisms of $N(M)_p$ spanned by the set $\{R_p^{\perp}(X, Y): X, Y \in T_pM\}$ coincides with that spanned by the set $\{\mathscr{R}_p^{\perp}(\xi, \eta): \xi, \eta \in N(M)_p\}$.*

In order to prove the last fact we shall next define $\mathscr{R}^{\perp}$ in an equivalent but convenient way.

First of all observe that, by (i) of Lemma 2.1, we can see, for each $p \in M$,

$$\mathscr{R}_p^{\perp}: \Lambda^2(N(M)_p) \to \mathscr{A}(N(M)_p)$$

by putting

$$\mathscr{R}_p^{\perp}(\xi \wedge \eta) = \mathscr{R}_p^{\perp}(\xi, \eta).$$

Similarly we can see

$$R_p^{\perp}: \Lambda^2(T_pM) \to \mathscr{A}(N(M)_p).$$

Consider now

$$\Lambda^2(N(M)_p) \xrightarrow{L_p} \mathscr{A}(T_pM) \xrightarrow{h_p} \Lambda^2(T_pM) \xrightarrow{R_p^{\perp}} \mathscr{A}(N(M)_p),$$

where $L_p(\xi \wedge \eta) = [A_\xi, A_\eta]$ and h_p is the isomorphism given by

$$\langle h_p^{-1}(x \wedge y)(u), v\rangle = \langle x, u\rangle \cdot \langle y, v\rangle - \langle y, u\rangle \cdot \langle x, v\rangle.$$

By a straightforward calculation we have

Lemma 2.3. $\mathscr{R}_p^\perp = -R_p^\perp \circ h_p \circ L_p$.

Put on $\Lambda^2(T_pM)$ the inner product defined by

$$((v, w)) = (h_p^{-1}(v), h_p^{-1}(w)) = -\operatorname{trace}(h_p^{-1}(v) \circ h_p^{-1}(w)).$$

Then we have

Lemma 2.4. $\ker R_p^\perp = (h_p \circ L_p(\Lambda^2(N(M)_p)))^\perp$.

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM, and let $u = \sum_{k<l} a_{kl} \cdot e_k \wedge e_l = \Lambda^2(T_pM)$. If $\xi, \eta \in N(M)_p$ are arbitrary,

$$\begin{aligned}\langle R_p^\perp(u)\xi, \eta\rangle &= \left\langle \sum_{k<l} a_{kl} \cdot R_p^\perp(e_k, e_l)\xi, \eta \right\rangle \\ &= \sum_{k<l} a_{kl} \cdot \langle [A_\xi, A_\eta](e_k), e_l\rangle\end{aligned}$$

but

$$\begin{aligned}((u, h_p \circ L_p(\xi \wedge \eta))) &= -\operatorname{trace}(h_p^{-1}(u) \circ L_p(\xi \wedge \eta)) \\ &= \sum_{s,t} \langle h_p^{-1}(u)(e_s), e_t\rangle \cdot \langle [A_\xi, A_\eta](e_s), e_t\rangle \\ &= 2 \cdot \sum_{k<l} a_{kl} \langle [A_\xi, A_\eta](e_k), e_l\rangle,\end{aligned}$$

then

$$\langle R_p^\perp(u)\xi, \eta\rangle = \tfrac{1}{2} \cdot ((u, h_p \circ L_p(\xi \wedge \eta))),$$

which clearly implies the Lemma. □

Proof of the proposition. It is immediate from Lemma 2.4. □

3. The main result

Theorem 3.1. *Let M^n be an immersed submanifold of a Riemannian manifold Q^N of constant curvature. Let $p \in M$ and let Φ^* be the restricted holonomy group of the normal connection at p. Then Φ^* is compact, there exists a unique (up to order) orthogonal decomposition of the normal space at p, $N(M)_p = \mathbf{V}_0 \oplus \cdots \oplus \mathbf{V}_k$, into Φ^*-invariant subspaces, and there exist $\Phi_0, \dots, \Phi_k$ normal Lie subgroups of Φ^* such that:*

(i) $\Phi^* = \Phi_0 \times \cdots \times \Phi_k$ *(direct product).*
(ii) Φ_i *acts trivially on* $\mathbf{V}_j$ *if* $i \neq j$.
(iii) $\Phi_0 = \{1\}$ *and, if* $i \geq 1$, Φ_i *acts irreducibly on* $\mathbf{V}_i$ *as the isotropy representation of a simple Riemannian symmetric space.*

We keep the notation and assumptions of §2.

Let $p \in M$ be fixed, and let $\gamma\colon [0, 1] \to M$ be a piecewise differentiable curve with $\gamma(1) = p$. Denote by $\gamma^*(\mathscr{R}^\perp)$ the tensor of type $(1, 3)$ in $N(M)_p$ defined by

$$\gamma^*(\mathscr{R}^\perp)(v, w)z = P_\gamma(\mathscr{R}_q^\perp(P_\gamma^{-1}(v), P_\gamma^{-1}(w))P_\gamma^{-1}(z)),$$

where $\gamma(0) = q$ and P_γ denotes the parallel displacement along γ with the normal connection.

Denote by $\mathscr{S}$ the linear subspace of the tensors of type $(1, 3)$ of $N(M)_p$, generated by all the $\gamma^*(\mathscr{R}^\perp)$, where γ runs over all piecewise differentiable curves ending at p.

From the theorem of Ambrose–Singer and Proposition 2.2 we have that the Lie algebra $\mathscr{g}$ of the restricted normal holonomy group Φ^* at p coincides with the linear span of the set $\{R(u, v)\colon R \in \mathscr{S},\ u, v \in N(M)_p\}$.

From Lemma 2.1, we have that if $R \in \mathscr{S}$, then

(i) $R(u, v) = -R(v, u)$,
(ii) $R(u, v)w + R(v, w)u + R(w, u)v = 0$,
(iii) $\langle R(u, v)w, z\rangle = -\langle w, R(u, v)z\rangle$,
(iv) $\langle R(u, v)w, z\rangle = \langle R(w, z)u, v\rangle$.

Decompose orthogonally

$$N(M)_p = \mathbf{V}_0 \oplus \cdots \oplus \mathbf{V}_k$$

into Φ^*-invariant subspaces such that Φ^* acts trivially in $\mathbf{V}_0$ and, if $i \geq 1$, Φ^* acts irreducibly in $\mathbf{V}_i$ $(\dim \mathbf{V}_i \geq 2)$.

If $u \in N(M)_p$, denote by u_i the projection of u into $\mathbf{V}_i$.

Lemma 3.2. *Let $x, y \in N(M)_p$ and let $R \in \mathscr{S}$. Then*

(i) $R(x_i, y_j) = 0$ *if* $i \neq j$;
(ii) $R(x, y) = \sum_{i=0}^k R(x_i, y_i)$;
(iii) $R(x_i, y_i)\mathbf{V}_j = \{0\}$ *if* $i \neq j$;
(iv) $R(x_i, y_i)\mathbf{V}_i \subset \mathbf{V}_i$.

Proof. It is the same of that given in [S, pp. 217, 218], but as it is very easy we write it down.

If $i \neq j$ and $u, v \in N(M)_p$ then

$$\langle R(x_i, y_j)u, v\rangle = \langle R(u, v)x_i, y_j\rangle = 0$$

because $R(u, v) \in \mathscr{g}$ and $\mathscr{g}$ leaves $\mathbf{V}_i$ invariant. This proves (i) and therefore (ii).

Let $v_j \in \mathbf{V}_j$, then

$$R(x_i, y_i)v_j = -R(y_i, v_j)x_i - R(v_j, x_i)y_i = 0$$

by part (i), which proves (iii).

Part (iv) is immediate. □

Let $\mathscr{g}_i$ be the vector subspace of $\mathscr{g}$ generated by all the $R(x_i, y_i)$, $R \in \mathscr{S}$ and $x_i, y_i \in \mathbf{V}_i$. From the above lemma we easily derive (see [S, p. 218]).

Lemma 3.3.

(i) $\mathscr{G}_0 = \{0\}$ *and each* $\mathscr{G}_i$ *is an ideal of* $\mathscr{G}$, *for* $i = 0, \ldots, k$;
(ii) $\mathscr{G} = \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_k$, *with* $[\mathscr{G}_i, \mathscr{G}_j] = \{0\}$ *if* $i \neq j$;
(iii) $\mathscr{G}_i \mathbf{V}_i = \mathbf{V}_i$ *for* $i = 0, \ldots, k$;
(iv) $\mathscr{G}_i \mathbf{V}_j = \{0\}$ *if* $i \neq j$;
(v) $\mathscr{G}_i$ *acts irreducibly in* $\mathbf{V}_i$, *for* $i = 1, \ldots, k$.

Proof of Theroem 3.1. We keep the notation of this section. For $i = 0, \ldots, k$, let Φ_i be the connected Lie subgroup of Φ^* (which is also connected) with Lie algebra $\mathscr{G}_i$. Lemma 3.3 implies that we have the direct product $\Phi^* = \Phi_0 \times \cdots \times \Phi_k$, that Φ_i acts trivially on $\mathbf{V}_j$ if $i \neq j$ and that Φ_i acts irreducibly on $\mathbf{V}_i$ if $i \geq 1$. The uniqueness part of the theorem is now clear.

Now, a connected Lie subgroup of orthogonal transformations of a vector space which acts irreducibly on it must be compact (see [K-N, appendix 5]). Then each Φ_i is compact, and therefore Φ^* is compact.

Now, for $i \geq 1$, choose $R_i \in \mathscr{S}$ such that R_i is not identically zero in $\mathbf{V}_i^3$. Each $[\mathbf{V}_i, R_i, \Phi_i]$ is an irreducible holonomy system, in the notation of [S]. Using [S, Theorem 5], we finish the proof, since, by Lemma 2.1(iv), R_i must have negative scalar curvature. □

In a future paper we will use the above results to establish the relation between isoparametric submanifolds and the sense of Terng and those in the sense of Strübing.

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