A LUECKING-TYPE SUBSPACE OF $\mathcal{L}_a^1$ AND ITS DUAL

PRATIBHA GHATAGE AND SHUNHUA SUN

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ABSTRACT. The purpose of this investigation is to determine the extent to which Luecking's decomposition of Bergman spaces $\mathcal{L}_a^p$, ($1 < p < \infty$) [4] can be extended to $\mathcal{L}_a^1$. The set of functions for which an atomic decomposition (using the reproducing kernel of the Bergman space) is possible turns out to be only a small part of $\mathcal{L}_a^1$. In this note we equip each of such functions with a new norm and study the resulting Banach space. We describe its dual and predual.

1. NOTATION

The Bergman space $\mathcal{L}_a^p$ ($1 \leq p < \infty$) consists of analytic functions for the open unit disk $\mathbb{D}$ whose p th powers are integrable with respect to dA , the normalized area measure of $\mathbb{D}$. Of particular interest to us is the space $\mathcal{L}_a^1 = \{f, f: \mathbb{D} \rightarrow \mathbb{C}, \int_{\mathbb{D}} |f(z)| dA(z) < \infty\}$ and its reproducing kernel $k_\alpha(z) = 1/(1 - \bar{\alpha}z)^2$, where $\alpha \in \mathbb{D}$. For more details see [2, p. 2].

If $f(z) = \sum_{n=1}^{\infty} a_n k_{\lambda_n}(z)$, where $\{\lambda_n\} \subseteq \mathbb{D}$ and $\sum |a_n| \|k_{\lambda_n}\|_1 < \infty$, then clearly $f \in \mathcal{L}_a^1$ and $\|f\|_1 \leq \sum_{n=1}^{\infty} |a_n| \|k_{\lambda_n}\|_1$. If we write

$$\|f\|_* = \inf \sum_{n=1}^{\infty} |a_n| \times \|k_{\lambda_n}\|_1,$$

where the infimum is taken over the set of all such decompositions of f , then $\|f\|_* \geq \|f\|_1$, $\|k_\lambda\|_* = \|k_\lambda\|_1$ for all λ and hence Y = the completion of the set of such functions in the $\|\cdot\|_*$ norm is a subspace of $\mathcal{L}_a^1$. We call $X = \{f, f: \mathbb{D} \rightarrow \mathbb{C}$ analytic, and $\sup_{|\alpha| < 1} |f(\alpha)| \|k_\alpha\|_1^{-1} < \infty\}$. Note that X is complete in the norm $\|f\|_{**} = \sum_{\alpha \in \mathbb{D}} |f(\alpha)| \|k_\alpha\|_1^{-1}$ and $X_0 = \{f \in X, \lim_{|\alpha| \rightarrow 1} |f(\alpha)| \|k_\alpha\|_1^{-1} = 0\}$ is a closed subspace of X . (In both cases, the proof is perfectly standard, and hence it is omitted.) The Bloch space $\mathcal{B}$

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consists of analytic functions $\{g, g: \mathbb{D} \rightarrow \mathbb{C}, \sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| < \infty\}$, and the little Bloch space $\mathcal{B}_0$ consists of those functions in $\mathcal{B}$ for which $\lim_{|z| \rightarrow 1} (1 - |z|^2) |g'(z)| = 0$ [1]. We conclude this section with the remark that as a simple consequence of the well-known fact that $(\mathcal{L}_a^1)^* = \mathcal{B}$, it follows that whenever $g \in \mathcal{B}$ we have $g(\alpha) = \langle g, k_\alpha \rangle$ and hence $|g(\alpha)| \|k_\alpha\|_1^{-1} \leq \|g\|_{\mathcal{B}}$. In other words, $\mathcal{B} \subseteq X$ and $\|g\|_{**} \leq \|g\|_{\mathcal{B}}$ whenever $g \in \mathcal{B}$. It is also true that $\mathcal{B}_0 \subseteq X_0$, but the proof requires a little more detail and we defer it to the next section.

2.

In this section we start out by giving an intrinsic description of Y . The description turns out not to be particularly surprising, but it enables us to show that Y is the dual of X_0 and predual of X . This in turn allows us to conclude that X_0 is indeed the closure of $\mathcal{B}_0$ (and hence of polynomials) in the $\|\cdot\|_{**}$ norm on analytic functions on the disk.

Proposition 1. $Y = \{f(z) = \int_{\mathbb{D}} k_\alpha(z) d\mu(\alpha), \text{ where } \mu \text{ is a Borel measure on } \mathbb{D} \text{ with } \|\mu\|(\partial\mathbb{D}) = 0 \text{ and } \int_{\mathbb{D}} \|k_\alpha\|_1 d|\mu|(\alpha) < \infty\}$ with $\|f\|_* = \inf_{\mu} \{\inf_{\mathbb{D}} \|k_\alpha\|_1 \times d|\mu|(\alpha)\}$, where the infimum is taken over all such decompositions of f .

The only nontrivial implication that any function $f(z)$ of the form specified in the proposition can be approximated in the $\|\cdot\|_*$ norm by discrete sums follows from standard measure-theoretic arguments. We sketch the proof for completeness.

The topological argument in our proof is based on the Whitney covering lemma [7, I, Theorem 3, p. 16, VI, Theorem 1, p. 167].

Lemma 0. Let F be a nonempty closed set in $\mathbb{R}^n$. Then its complement Ω is the union of a sequence of rectangles Q_k whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F .

Corollary 1. Given $\epsilon > 0$ there exists a family of disks $\{D_k\}_{k=1}^\infty$, $D_k = D_k(\lambda_k, r_k)$, and a disk centred at λ_k and having radius r_k , satisfying the following conditions:

- (1) $D_k \subseteq \mathbb{D}$, $|\mathbb{D} \setminus \bigcup_{k=1}^\infty D_k| = 0$, where $|\cdot|$ denotes area measure.
- (2) $D_k \cap D_{k'} = \emptyset$ if $k \neq k'$
- (3) $\text{diam } D_k = 2r_k \leq \min\{\epsilon \text{ dist}(D_k, \partial\mathbb{D})\}$.

Proof. We apply the covering lemma recursively, starting with $F = \mathbb{R}_2 \setminus \mathbb{D}$. Subdividing the rectangles further if necessary, we may choose a family of rectangles $\{R_1, R_2, \dots\}$ with disjoint interiors, so that they fill up $\mathbb{D}$ and $\text{diam } R_k \leq \min\{\epsilon, \text{dist}(R_k, \partial\mathbb{D})\}$. We now subdivide each rectangle R_k by means of disks in the following simple way; we have a family of disks $\{D_{k,\ell}^{(1)}\}_{\ell=1}^\infty \subseteq R_k$, $|\bigcup_{\ell=1}^\infty D_{k,\ell}^{(1)}| > \frac{1}{4}|R_k|$, as shown in Figure 1.

FIGURE 1.

By successive applications of the lemma to $\mathbb{R}^2 \setminus \bigcup_{j=1}^n \bigcup_{k,\ell} D_{k,\ell}^{(j)}$ we obtain a countable family of open disks $\{D_{k,\ell}^{(j)} : k, \ell, j \geq 1\}$ such that $|\mathbb{D} \setminus \bigcup_{j=1}^n \bigcup_{k,\ell} D_{k,\ell}^{(j)}| < (\frac{3}{4})^n \pi$. Simply reenumerating them, we have a countable family of disks $\{D_k\}$ which satisfy the three conditions stated above.

Lemma 1. *If f belongs to the disk algebra $A(\mathbb{D})$, then for any $\epsilon > 0$ there exists sequences $\{a_n\}_{n=1}^\infty \subseteq \mathbb{C}$ and $\{\lambda_n\} \subseteq \mathbb{D}$ such that $\sum_{n=1}^\infty |a_n| \|k_{\lambda_n}\|_1 < \infty$ and $\|f - \sum_{n=1}^\infty a_n k_{\lambda_n}\|_* < \epsilon$.*

Proof. Elementary but central to our proof is the computation of $\|k_\alpha\|_1$ where k_α is the reproducing kernel for the Bergman space. Note that

$$\|k_\alpha\|_1 = \langle 1/1 - \bar{\alpha}z, 1/1 - \bar{\alpha}z \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathcal{L}_a^2(\mathbb{D})$. So

$$\|k_\alpha\|_1 = \left\| \sum_{n \geq 0} \bar{\alpha}^n z^n \right\|_{\mathcal{L}_a^2}^2 = \sum_{n \geq 0} |\alpha|^{2n} \|z^n\|^2 = \sum_{n \geq 0} \frac{|\alpha|^{2n}}{n+1},$$

and

$$(1) \quad \|k_\alpha\|_1 \sum_{n \geq 0} \frac{|\alpha|^{2n}}{n+1} = |\alpha|^{-2} \ln(1 - |\alpha|^2)^{-1}, \quad |\alpha| \neq 0,$$

Consequently,

$$M = \int_{\mathbb{D}} \|k_\alpha\|_1 dA(\alpha) = \sum_{n \geq 0} \frac{1}{(n+1)(2n+1)} < \infty.$$

Now let $f \in A(\mathbb{D})$. For given $\epsilon > 0$ there exists $\delta > 0$ such that $z_1, z_2 \in \mathbb{D}$ and $|z_1 - z_2| < \delta$ implies $|f(z_1) - f(z_2)| < \epsilon/M$. For this ϵ we choose a family of disks D_k centred at λ_k and having radius r_k satisfying the three conditions stated in Corollary 1. Note that $f \in \mathcal{L}_a^2(\mathbb{D})$, and hence $f(z) = \int_{\mathbb{D}} f(\alpha) k_\alpha(z) dA(\alpha) = \sum_{k=1}^\infty \int_{D_k} k_\alpha(z) f(\alpha) dA(\alpha)$ (by condition (1) of the previous corollary) $= \sum_{k=1}^\infty \int_{D_k} k_\alpha(z) (f(\alpha) - f(\lambda_k)) dA(\alpha) + \sum_{k=1}^\infty f(\lambda_k) \int_{D_k} k_\alpha(z) dA(\alpha)$. The second term clearly $= \sum_{k=1}^\infty (f(\lambda_k) r_k^2) k_{\lambda_k}(z)$.

If we write $a_k = f(\lambda_k)r_k^2$, then

$$f(z) - \sum_{k=1}^{\infty} a_k k_{\lambda_k}(z) = \sum_{k=1}^{\infty} \int_{D_k} (f(\alpha) - f(\lambda_k)) k_{\alpha}(z) dA(\alpha).$$

First we show that $\sum_{k=1}^{\infty} |a_k| \|k_{\lambda_k}\|_1 < \infty$. Observe that by condition (3) of Corollary 1 we may choose constants C_1 and C_2 ($C_1, C_2 > 0$, $C_1 < 1$, $C_2 > 1$) such that

$$C_1 < \frac{1 - |\lambda_k|^2}{1 - |z|^2} < C_2$$

for $z \in D_k$, whenever $k \geq 1$. It follows that $|\ln(1 - |\lambda_k|^2)| \leq |\ln(1 - |z|^2)| + C$ for $z \in D_k$, $k \geq 1$. Hence

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k| \|k_{\lambda_k}\|_1 &= \sum_{k=1}^{\infty} |f(\lambda_k)| r_k^2 |\lambda_k|^{-2} |\ln(1 - |\lambda_k|^2)| \\ &\leq \|f\|_{\infty} \left[\sum_{k \geq 1, |\lambda_k| < \frac{1}{2}} r_k^2 |\lambda_k|^{-2} |\ln(1 - |\lambda_k|^2)| + \sum_{k \geq 1, |\lambda_k| \geq \frac{1}{2}} r_k^2 |\lambda_k|^{-2} |\ln(1 - |\lambda_k|^2)| \right] \\ &\leq \|f\|_{\infty} \left[\sup_{0 < x < 1/2} \left| \ln \frac{1 - x^2}{x^2} \right| \sum_{k=1}^{\infty} r_k^2 + 4 \sum_{k=1}^{\infty} \int_{D_k} (|\ln(1 - |z|^2)| + C) dA(z) \right] \\ &< \infty. \end{aligned}$$

It remains to show that

$$g(z) = \sum_{k=1}^{\infty} \int_{D_k} (f(\alpha) - f(\lambda_k)) k_{\alpha}(z) dA(\alpha) \in Y \quad \text{and} \quad \|g\|_* < \epsilon.$$

Note $g(z) = \int_{\mathbb{D}} k_{\alpha}(z) d\mu(\alpha)$ where $d\mu(\alpha) = \sum_{k \geq 1} (f(\alpha) - f(\lambda_k))_{\chi_{D_k}}(\alpha) dA(\alpha)$. Hence $\int_{\mathbb{D}} \|k_{\alpha}\|_1 d|\mu|(\alpha) \leq \frac{\epsilon}{M} \int_{\mathbb{D}} \|k_{\alpha}\|_1 dA(\alpha) = \epsilon$. We remark that, as a trivial consequence of this argument, we have $A(\mathbb{D}) \subseteq Y$.

The next lemma shows that $A(\mathbb{D})$ is in fact dense in Y .

Lemma 3. Suppose $f(z) = \int_{\mathbb{D}} k_{\alpha}(z) d\mu(\alpha) \in Y$ and, for $\delta > 0$, let $f_{\delta}(z) = \int_{|z| \leq 1-\delta} k_{\alpha}(z) d\mu(\alpha)$. Then $f_{\delta} \in A(\mathbb{D})$ and $\|f - f_{\delta}\|_* \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. $f(z) - f_{\delta}(z) = \int_{1-\delta < |z| < 1} k_{\alpha}(z) d\mu(\alpha)$. Clearly $f_{\delta} \in Y$ and $\|f - f_{\delta}\|_* \leq \int_{1-\delta < |z| < 1} \|k_{\alpha}\|_1 d|\mu|(\alpha)$. As $|\mu|\{z, 1 - \delta < z < 1\} \rightarrow 0$ and $\|k_{\alpha}\|_1 \in \mathcal{L}^1(d|\mu|)$ by choice of μ , we have $\|f - f_{\delta}\|_* \rightarrow 0$ as $\delta \rightarrow 0$. In order to show that $f_{\delta} \in A(\mathbb{D})$, simply note that $k_{\alpha}(z) = \sum_{n \geq 0} (n+1) \bar{\alpha}^n z^n$ and

$$\begin{aligned} |f_{\delta}(z) - \sum_{n=0}^N \left(\int (n+1) \bar{\alpha}^n d\mu(\alpha) \right) z^n| &\leq \sum_{N+1}^{\infty} \int_{|\alpha| \leq 1-\delta} (n+1) |\alpha|^n d|\mu|(\alpha) \\ &\leq C \sum_{n=N+1}^{\infty} (n+1) (1-\delta)^n \end{aligned}$$

which is the tail-end of a series converging to $1/\delta(1-\delta)$ and $C = \int_{|\alpha| \leq 1-\delta} d|\mu|(\alpha)$. Thus f_δ is a uniform limit of polynomials and hence belongs to $A(\mathbb{D})$.

Proposition 2. *The dual of X_0 can be identified with Y . More precisely, every bounded linear functional on X_0 is the form $h \mapsto \langle h, f \rangle = \int_{\mathbb{D}} h(z) \bar{f}(z) dA(z)$ for a unique $f \in Y$, and the norm of the linear functional on X_0 induced by f is equivalent to $\|f\|_*$.*

Proof. Suppose $f(z) = \int_{\mathbb{D}} k_\alpha(z) d\mu(\alpha) \in Y$ and $h \in X$. Note that if $h \in X$ and $\alpha \in \mathbb{D}$ is fixed, $|h(z)k_\alpha(z)| \leq C|h(z)| \leq C\|k_z\|_1$ and, as shown at the beginning of the proof of Lemma 1 $\|k_z\|_1 \in \mathcal{L}^1(d|\mu|)$. Hence by the standard argument $h(\alpha) = \langle h, k_\alpha \rangle = \int_{\mathbb{D}} h(z) \bar{k}_\alpha(z) dA(z)$. Now

$$\begin{aligned} \langle h, f \rangle &= \int_{\mathbb{D}} \int_{\mathbb{D}} h(z) \bar{k}_\alpha(z) dA(z) d\bar{\mu}(\alpha) \\ &= \int_{\mathbb{D}} h(\alpha) d\bar{\mu}(\alpha), \end{aligned}$$

and we have $|\langle h, f \rangle| \leq \int_{\mathbb{D}} \|h\|_{**} \|k_\alpha\|_1 d|\mu|(\alpha)$ for all μ which represent f . Hence $|\langle h, f \rangle| \leq \|h\|_{**} \|f\|_*$. Conversely, suppose that L is a bounded linear functional on X_0 . Define $S: X_0 \rightarrow C(\mathbb{D})$ by $Sh(\alpha) = h(\alpha) \|k_\alpha\|_1^{-1}$ if $\alpha \in \mathbb{D}$ and $Sh(\alpha) = 0$ if $\alpha \in \partial\mathbb{D}$, and let $L_1(g) = L \circ S^{-1}(g)$ whenever $g \in S(X_0)$. Note that by definition of X_0 and $\|\cdot\|_{**}$, S is an isometry and hence $\|L_1\| = \|L\|$. By the standard applications of Hahn–Banach and Riesz representation theorems, we have a measure μ on $\mathbb{D}$ (cutting it down to $\mathbb{D}$ if necessary) such that $L(h) = \int_{\mathbb{D}} \|k_\alpha\|_1^{-1} h(\alpha) d\mu(\alpha)$ and $|\mu|(\mathbb{D}) \leq \|L\|$. If we define $f(z) = \int_{\mathbb{D}} k_\alpha(z) \|k_\alpha\|_1^{-1} d\bar{\mu}(\alpha)$, then $f \in Y$, $\|f\|_* \leq \|L\|$, and $L(h) = \int_{\mathbb{D}} h(z) \bar{f}(z) dA(z)$. Uniqueness follows from the fact that f is analytic and X_0 contains polynomials.

Corollary 2. *X_0 is the closure of the little Bloch space $\mathcal{B}_0$ (in fact, of polynomials) in the $\|\cdot\|_{**}$ norm.*

Proof. If L is in X_0^* and $L(z^n) = 0$ for $n \geq 0$, then the unique function $f \in Y$ which represents L is clearly identically zero. This shows that X_0 is the closure of polynomials, and hence of $\mathcal{B}_0$.

Remark 1. $\mathcal{B}_0$ is clearly a proper subset of X_0 . The easiest way to see this is to note that $\|k_\alpha\|_1^{-1} \rightarrow 0$ as $|\alpha| \rightarrow 1$, and hence $H^\infty(\mathbb{D}) \subseteq X_0$. However there exists an infinite Blaschke product which does not belong to $\mathcal{B}_0$ [6]. A natural question is whether the Bloch space $\mathcal{B}$ is dense in X . That remains open.

Corollary 3. *The unit ball of $Y = (Y)_1 = \Gamma^{-w^*}$ where*

$$\Gamma = \left\{ \sum c_i k_{\alpha_i} / \|K_{\alpha_i}\|_1, \sum |c_i| \leq 1 \right\}$$

Proof. Recall that $S: X_0 \rightarrow C_0(\mathbb{D})$ is an isometry from X_0 into the set of continuous functions on $\mathbb{D}$ which vanish on $\partial\mathbb{D}_*$ and hence $S^*: C_0(\mathbb{D})^* \rightarrow X_0^* = Y$ is a partial isometry onto Y . It is well known that $((C_0(\mathbb{D})^*)^*)_1$ is

wk^{-} closure of $\{\sum_i c_i \delta_{\alpha_i}, \sum |c_i| \leq 1\}$. To complete the proof of the corollary we note that $\langle S^* \delta_\alpha, h \rangle = \langle \delta_\alpha, Sh \rangle = \langle k_\alpha / \|k_\alpha\|_1, h \rangle$ for all $h \in X_0$, and hence $S^* \delta_\alpha = k_\alpha / \|k_\alpha\|_1$.

Proposition 3. *The dual of Y can be identified with X . More precisely, every bounded linear functional on Y is of the form $f \mapsto \langle f, h \rangle = \int_{\mathbb{D}} f(z) \bar{h}(z) dA(z)$ for a unique $h \in X$, and the norm of the linear functional induced by h is equivalent to $\|h\|_{**}$.*

Proof. It is clear from the first part of the proof of Proposition 2 that any function $h \in X$ induces a bounded linear functional L on Y and $\|L\| \leq \|h\|_{**}$.

Conversely, suppose that L is a bounded linear functional on Y , and define $h(\alpha) = \bar{L}(k_\alpha)$ for $\alpha \in \mathbb{D}$. Note that $\|k_\alpha\|_* = \|k_\alpha\|_1$; hence $\sup_{\alpha \in \mathbb{D}} \{\|k_\alpha\|_1^{-1} |h(\alpha)|\} \leq \|L\|$. Next we prove that h is analytic. As a first step in this proof we claim that h is continuous. Fix $\alpha \in \mathbb{D}$ and choose a positive $r < (1 - |\alpha|/2)$. If $D(\alpha, r)$ is the Euclidean disk of radius r centred at α , then $\bar{L}(\int_{D(\alpha, r)} k_\lambda(z) dA(\lambda)) = r^2 h(\alpha)$. To see this, let $\rho = \bar{\lambda} - \bar{\alpha}$ and write

$$k_\lambda(z) = \frac{1}{(1 - \bar{\lambda}z)^2} = \frac{1}{(1 - \rho z - \bar{\alpha}z)^2} = \frac{1}{(1 - \bar{\alpha}z)^2} + \beta_1 \rho + \beta_2 \rho^2 + \dots;$$

hence, $\bar{L}(\int_{D(\alpha, r)} k_\lambda(z) dA(\lambda)) = \bar{L}(\int_{D(\alpha, r)} k_\lambda(z) dA(\lambda)) = \bar{L}(r^2 k_\alpha(z)) = r^2 h(\alpha)$. Now if $\beta \in D$, choose r such that $0 < r < (1 - |\alpha|)/2, (1 - |\beta|)/2$, and write $D_1 = D(\alpha, r), D_2 = D(\beta, r)$. Then

$$\begin{aligned} r^2(h(\alpha) - h(\beta)) &= \bar{L} \left(\int_{D_1} k_\lambda(z) dA(\lambda) - \int_{D_2} k_\lambda(z) dA(\lambda) \right) \\ &= \bar{L} \left(\int_{D_1 - D_2} k_\lambda(z) dA(\lambda) - \int_{D_2 - D_1} k_\lambda(z) dA(\lambda) \right) \\ &\Rightarrow r^2 |h(\alpha) - h(\beta)| \\ &\leq \|L\| \left(\left\| \int_{D_1 - D_2} k_\lambda(z) dA(\lambda) \right\|_* + \left\| \int_{D_2 - D_1} k_\lambda(z) dA(\lambda) \right\|_* \right) \\ &\leq \|L\| C |D_1 \Delta D_2| \end{aligned}$$

where $C = \sum \{\|k_\lambda\|_1, \lambda \in \mathbb{D}_1 \cup \mathbb{D}_2\}$ and $D_1 \Delta D_2$ is the usual symmetric difference. Clearly $h(\beta) \rightarrow h(\alpha)$ as $\beta \rightarrow \alpha$, and h is continuous. Now if Γ is a closed path lying inside $\mathbb{D}$, then $\int_\Gamma k_\lambda d\lambda = 0$ and $\sup\{\|k_\lambda\|_* = \|k_\lambda\|_1, \lambda \in \Gamma\} < \infty$. Hence $\int_\Gamma \bar{L}(k_\lambda) d\lambda = \int h(\lambda) d\lambda = 0$, and by Morera's theorem h is analytic. Thus $h \in X$.

It remains to show that h induces L . It was shown in Proposition 2 that $h(\alpha) = \int_{\mathbb{D}} h(z) \bar{k}_\alpha(z) dA(z) = \langle h, k_\alpha \rangle = \langle \bar{k}_\alpha, h \rangle$, so by definition of $h(\alpha)$ we have $\bar{L}(k_\alpha) = \langle k_\alpha, h \rangle$. If $f(z) = \sum_{n=1}^\infty a_n k_{\lambda_n}$, $\sum_{n \geq 1} |\alpha_n| \|k_{\lambda_n}\|_1 < \infty$, then $L(f) = \sum_{n=1}^\infty a_n \langle k_{\lambda_n}, h \rangle$ by continuity of L . As we know that $h \in X$ implies

that $g \mapsto \langle g, h \rangle$ is continuous on Y , $\langle f, h \rangle = \sum_{n \geq 1} a_n \langle k_{\lambda_n}, h \rangle$, and hence $L(f) = \langle f, h \rangle$. By Proposition 1 discrete sums like f are dense in Y and the proof is complete.

Remark 2. One trivial consequence of this is to show that $Y \neq \mathcal{L}_\alpha^1(dA)$. If this were the case, then we would have $X = \mathcal{B}$, making the two norms equivalent. This in turn would imply that $\mathcal{B}_0 = X_0$, which was shown to be false. (See Remark 1). Another standard consequence of this duality is that X_0 is wk^* -dense in X . It now follows from Corollary 3 that $\mathcal{B}_0$ and hence $\mathcal{B}$ are wk^* -dense in X . As remarked earlier, we have not been able to determine whether $\mathcal{B}$ is norm-dense in X . Also, is $X_0 \subseteq \mathcal{B}$?

As a consequence of this proposition, we now have a stronger version of Corollary 3.

Corollary 4. The unit ball of $Y = (Y)_1 = \bar{\Gamma}$ where

$$\Gamma = \left\{ \sum_{i=1}^n c_i k_{\alpha_i} / \|k_{\alpha_i}\|_1, \sum |c_i| \leq 1 \right\},$$

and hence

$$Y = \left\{ \sum_i a_i k_{\alpha_i}, \sum_i |a_i| \|k_{\alpha_i}\|_1 < \infty \right\};$$

i.e., the discrete sums are complete with respect to the norm $\|\cdot\|_*$.

Proof. Suppose $f \in (Y)_1$. Without loss of generality, we may write $f(z) = \int_D k_\alpha(z) d\mu_f(\alpha)$ where $\int_D \|k_\alpha\|_1 d|\mu_f|(\alpha) \leq 1$. We claim that whenever $L \in Y^*$, $|L(f)| \leq \sup_\alpha |L(k_\alpha / \|k_\alpha\|_1)|$. By Proposition 3, and $L \in Y^*$ is induced by a function $h \in X$; $L(\int_D g(z) \bar{h}(z) dA(z))$ and $\|L\| \sim \|h\|_{**}$. In particular, $L(f) = \int_D \int_D k_\alpha(z) \bar{h}(z) dA(z) d\mu_f(\alpha) = \int_D L(k_\alpha) d\mu_f(\alpha)$, and hence $|L(f)| \leq \sup_\alpha |L(k_\alpha / \|k_\alpha\|_1)|$ since $\int_D \|k_\alpha\|_1 d|\mu_f|(\alpha) \leq 1$. This proves the claim. Now if there exists $f \in (Y)_1$ which does not belong to $\bar{\Gamma}$ then by the Hahn–Banach theorem we may choose $L \in Y^*$ such that $\operatorname{Re} L(f) > \sup\{\operatorname{Re} L(g), g \in \Gamma\}$. Note that $g \in \Gamma$ implies $e^{i\theta} g \in \Gamma$ for all real θ ; hence $\sup\{\operatorname{Re} L(g), g \in \Gamma\} = \sup\{|L(g)|, g \in \Gamma\}$. Thus $|L(f)| > \sup |L(k_\alpha) / \|k_\alpha\|_1|$, which is clearly a contradiction. Now if $f \in (Y)$, write $f = f_1 + g_1$, $\|f_1\|_* \leq \frac{1}{2}$, $g_1 \in \Gamma$, and note that $2f_1 \in (Y)_1$. By induction, we obtain a sequence of g_i 's in Γ such that $f = \sum_{i=1}^\infty 2^{-i+1} g_i$.

Remark 3. It is also natural to ask how the space Y relates to $\mathcal{L}_\alpha^1(d\nu)$ where ν is the weighted area measure given by $d\nu(z) = \|k_z\|_1 dA(z)$. It is easy to see that any function $h \in \mathcal{L}_\alpha^1(d\nu)$ induces a bounded linear functional $f \mapsto \langle f, h \rangle$ on $f \in X_0$. This coupled with the duality in Proposition 2 clearly implies that $h \in Y$. Thus $\mathcal{L}_\alpha^1(d\nu) \subseteq Y \subseteq \mathcal{L}_\alpha^1(dA)$. In particular ν is not a Carleson measure in $\mathbb{D}$ [2, p. 26]. Thus, $\mathcal{L}_\alpha^1(d\nu) = Y$ is equivalent to the seemingly weaker condition $\sup_{\lambda \in \mathbb{D}} \|k_\lambda\|_{\mathcal{L}_\alpha^1(d\nu)} / \|k_\lambda\|_1 < \infty$. Note that ν being

a circular measure, it is easy to show $\|k_\lambda\|_{\mathcal{L}_n^1(d\nu)} = \sum_{n \geq 0} (\sum_{k \geq 1} \frac{1}{k(k+n)} |\lambda|^{2n})$, whereas $\|k_\lambda\|_1 = \sum_{n \geq 0} (n+1)^{-1} |\lambda|^{2n}$. As $\sum_{k \geq 1} [k(k+n)]^{-1} \sim (\log n)/n$, an elementary computation shows that $\sup_{\lambda \in \mathbb{D}} \|k_\lambda\|_{\mathcal{L}_n^1(d\nu)} / \|k_\lambda\|_1 = \infty$.

More generally, does there exist a positive measure λ on $\mathbb{D}$ s.t. $\|k_z\|_{\mathcal{L}_n^1(d\lambda)} \leq M \|k_z\|_1$ for all $z \in \mathbb{D}$ but λ is not a Carleson measure?

In conclusion, we remark that it is possible to define a proper subspace of $\mathcal{L}_\alpha^2$, suitably renormed so that functions in X (respectively X_0) induce continuous (completely continuous) Hankel operators on it. We intend to pursue this and related questions in a forthcoming paper.

Added in proof. The answer to our question at the end of Remark 3 is negative if λ is a circular measure [8, p. 257]. We thank the referee for bringing [8] to our notice.

REFERENCES

1. J. M. Anderson, *Bloch functions: the basic theory, operators and function theory* (S. C. Power, ed.), D. Reidel, Dordrecht, 1985, pp. 1–17.
2. S. Axler, *Bergman spaces and their operators*, in *Surveys of Some Recent Results in Operator Theory*, vol. 1 (J. B. Conway and B. B. Morrel, eds.), Pitman Research Notes in Math., vol. 171, 1988, pp. 1–50.
3. D. Luecking, *A technique for characterizing Carleson measures on Bergman spaces*, Proc. Amer. Math. Soc. **87** (1983), 656–660.
4. —, *Representation and duality in weighted spaces of analytic functions*, Indiana University Math. J. **34** (1985), 319–336.
5. R. Rochberg, *Decomposition theorems for Bergman spaces and their applications*, in *Operators and Function Theory* (S. C. Power, ed.), D. Reidel, Dordrecht, 1985, pp. 225–277.
6. D. Sarason, *Blaschke products in $\mathcal{B}_0$* , Linear and Complex Analysis Problem Book (V. P. Havin, S. V. Hrusčëv, and N. K. Nickol'skii, eds.), Lecture Notes in Math., vol. 1043, Springer, Berlin, 1984, pp. 337–338.
7. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.
8. A. Shields and D. Williams, *Bounded projections, duality, and multipliers in spaces of harmonic functions*, J. Reine Angew. Math. **299/300** (1978), 256–279.

DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND, OHIO 44115

DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, 610064 PEOPLES' REPUBLIC OF CHINA