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JONES AND Q POLYNOMIALS FOR 2-BRIDGE KNOTS AND LINKS

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Dedicated to Professor Yoko Tao on her sixtieth birthday

ABSTRACT. It is known that the Q polynomial of a 2-bridge knot or link can be obtained from the Jones polynomial. We construct arbitrarily many 2-bridge knots or links with the same Q polynomial but distinct Jones polynomials.

The Jones polynomial $V_L(t) \in Z[t^{\pm 1/2}]$ [5] is an invariant of the isotopy type of an oriented knot or link L in the 3-sphere. The writhe $w(D)$ of an oriented planar diagram D of L is the sum of the signs at all the crossings of D, according to the convention explained in Figure 1 (p. 836). Let $|D|$ be a diagram D with its orientation unknown. Then Kauffman's bracket polynomial $\langle D \rangle \in Z[A^{\pm 1}]$ [10] of $|D|$, which is a regular isotopy invariant, is defined by

$$\langle 0 \rangle = 1 \text{ for a simple closed curve } 0,$$
$$\langle D' \rangle = (-A^2 - A^{-2})\langle D \rangle,$$
$$\langle D_\pm \rangle = A^{\pm 1}\langle D_0 \rangle + A^{\mp 1}\langle D_\infty \rangle,$$

where $|D'|$ is the disjoint union of $|D|$ and a simple closed curve and $|D_i|$ are identical diagrams except near one point, where they are as in Figure 2 (p. 836). Then the Jones polynomial can be defined using the formula

$$V_L(A^4) = (-A^3)^{-w(D)}\langle D \rangle.$$

The Q polynomial $Q_L(x) \in Z[x^{\pm 1}]$ [1, 4] is an invariant of the isotopy type of an unoriented knot or link $|L|$ in the 3-sphere, which is defined by the following formulas:

$$Q_U(x) = 1 \text{ for the unknot } U,$$
$$Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_\infty}(x)),$$

where the links $|L_i|$ have diagrams $|D_i|$ which are as in the above.

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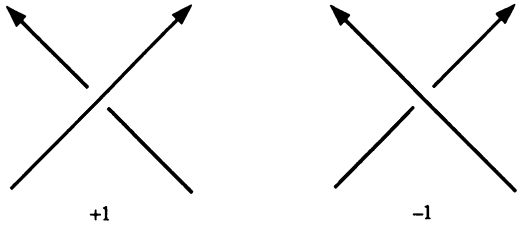

FIGURE 1

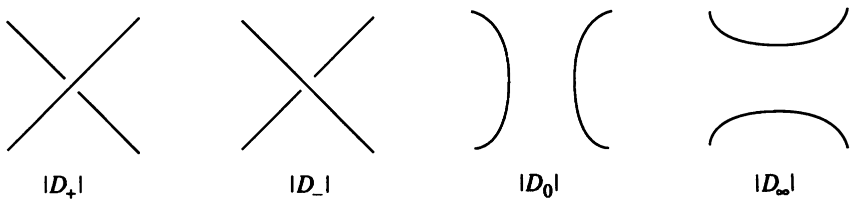

FIGURE 2

The author [7] discovered a relation between the Jones and Q polynomials of a 2-bridge knot or link (see, for example, [2, Chapter 12]).

Theorem *. *If L is a 2-bridge knot or link, then it holds that*

$$Q_L(x) = 2x^{-1}V_L(t)V_L(t^{-1}) + 1 - 2x^{-1},$$

where $x = -t - t^{-1}$. Identically,

$$Q_L(x) = 2x^{-1}\langle D\rangle\langle D!\rangle + 1 - 2x^{-1},$$

where $x = -A^4 - A^{-4}$, D is a diagram for L, and $D!$ is a mirror image of D, so that $\langle D!\rangle(A) = \langle D\rangle(A^{-1})$.

This theorem implies that the Q polynomial of a 2-bridge knot or link can be deduced from the Jones polynomial. Conversely, even if a Q polynomial of some 2-bridge knot or link is given, we cannot necessarily infer its Jones polynomial. In fact, through a computer calculation of polynomial invariants of 2-bridge knots and links [9], except for a reflection such as right- and left-handed trefoils, we found many pairs of 2-bridge knots and links with the same Q polynomial but distinct Jones polynomials; $\{10_{14}, 10_{31}\}$, $\{10_{19}, 10_{36}\}$, and $\{9_4^2, 9_{10}^2\}$ are such pairs in the table of [15]. Also it is known that there exist arbitrarily many skein-equivalent 2-bridge knots [8] and links [6], which thus have the same 2-variable Jones, Jones, Alexander, and Q polynomials. (See [12] for the definition of skein equivalence and 2-variable Jones polynomial.)

Generalizing these examples, we prove

Theorem. *For any positive integer N, there exist N sets of 2^N 2-bridge knots (resp. links) $\mathscr{K}_1, \mathscr{K}_2, \dots, \mathscr{K}_N$ with $\mathscr{K}_i = \{K_{i1}, K_{i2}, \dots, K_{i2^N}\}$ such that*

(i) *all the knots (resp. links) in the union $\bigcup_{i=1}^N \mathscr{K}$ share the same Q and Alexander (resp. 2-variable Alexander) polynomials;*
(ii) *all the knots (resp. links) in each $\mathscr{K}_i$ are skein-equivalent, and so they have the same 2-variable Jones, Jones, and Alexander (resp. reduced Alexander) polynomials; and*
(iii) *all the knots (resp. links) $K_{11}, K_{21}, \dots, K_{N1}$ have mutually distinct Jones polynomials.*

1. Preliminaries

Let α be a 3-braid. We denote a 3-braid $\alpha S_2^n \alpha^{-1}$ and $\alpha S_2^m \alpha^{-1} S_1^n \alpha$, m, $n \in Z$, by $\alpha(n)$ and $\alpha(m, n)$, respectively, where S_1 and S_2 are elementary 3-braids as shown in Figure 3. Let G_α and H_α be unoriented 2-bridge knot or link diagrams as shown in Figure 4. From [8, Proposition 2.4], we have

Lemma 1.

(i) $\langle H_{\alpha(n)} \rangle = A^n[d - \{1 - (-A^{-4})^n\} d^{-1} \langle G_\alpha \rangle \langle G_\alpha ! \rangle]$,

(ii)
$$\langle G_{\alpha(m,n)} \rangle = A^{m+n} \langle G_\alpha \rangle [(-A^{-4})^m + (-A^{-4})^n - 1 + \{1 - (-A^{-4})^m\} \times \{1 - (-A^{-4})^n\} d^{-2} \langle G_\alpha \rangle \langle G_\alpha ! \rangle,$$

where $d = -A^2 - A^{-2}$ is the bracket polynomial of a trivial 2-component link diagram without any crossing.

Let $Q_\alpha(x)$, $Q_{\alpha(n)}(x)$, and $Q_{\alpha(m,n)}(x)$ be the Q polynomials of unoriented 2-bridge knots or links with diagrams G_α, $H_{\alpha(n)}$, and $G_{\alpha(m,n)}$, respectively.

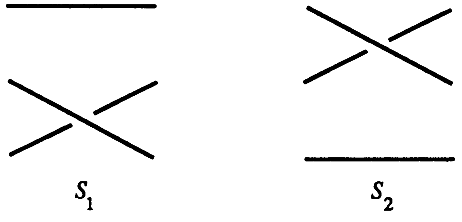

Figure 3

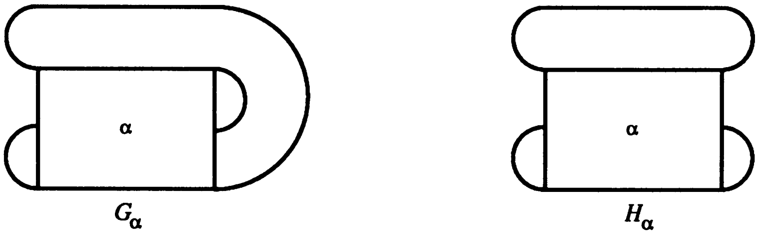

FIGURE 4

Then we have

Lemma 2.

(i) $Q_{\alpha(n)}(x) = \frac{1}{2}(\sigma_{n+1} - \sigma_{n-1})(\mu - \mu^{-1}Q_\alpha(x)^2) + \mu^{-1}Q_\alpha(x)^2$,

(ii) $Q_{\alpha(1,-1)}(x) = (x^2/4)(Q_\alpha(x)+1)^2(Q_\alpha(x)+\mu) - \mu$,

where $\sigma_n \in Z[x^{\pm 1}]$ *is the polynomial defined by* $\sigma_{n-1} + \sigma_{n+1} = x\sigma_n$, $\sigma_0 = 0$, $\sigma_1 = 1$, *and* $\mu = 2x^{-1} - 1$ *is the* Q *polynomial of a trivial* 2-*component link.*

Proof.

(i) is Lemma 6.1 of [6].

(ii) By Lemma 1 (ii),

$$\langle G_{\alpha(1,-1)}\rangle = \langle G_\alpha\rangle(-A^{-4} - 1 - A^4 + \langle G_\alpha\rangle\langle G_\alpha!\rangle).$$

By substituting $\langle G_\alpha\rangle\langle G_\alpha!\rangle = \frac{x}{2}(Q_\alpha(x)+\mu)$ (Theorem *) and $A^{-4} + A^4 = -x$, this becomes

$$\langle G_{\alpha(1,-1)}\rangle = \frac{x}{2}\langle G_\alpha\rangle(Q_\alpha(x)+1),$$

from which we have

$$\langle G_{\alpha(1,-1)}!\rangle = \frac{x}{2}\langle G_\alpha!\rangle(Q_\alpha(x)+1).$$

Substituting these formulas into $\langle G_{\alpha(1,-1)}\rangle\langle G_{\alpha(1,-1)}!\rangle = \frac{x}{2}(Q_{\alpha(1,-1)}(x)+\mu)$ (Theorem *), we obtain (ii).

Using Lemma 2, we can prove the following by induction:

Lemma 3.

$$\begin{aligned} Q_{\alpha(m,n)}(x) = {} & \frac{1}{4\mu^2}(\sigma_{m+1} - \sigma_{m-1} - 2)(\sigma_{n+1} - \sigma_{n-1} - 2)Q_\alpha(x)^3 \\ & - \frac{1}{4}x(x+2)\sigma_m\sigma_n Q_\alpha(x)^2 \\ & + \left\{1 - \frac{1}{4}(\sigma_{m+1} - \sigma_{m-1} - 2)(\sigma_{n+1} - \sigma_{n-1} - 2)\right\} Q_\alpha(x) \\ & + (\mu^2/4)x(x+2)\sigma_m\sigma_n. \end{aligned}$$

Since $\sigma_{n+1} - \sigma_{n-1} = \sigma_{-n+1} - \sigma_{-n-1}$ and $\sigma_m\sigma_n = \sigma_{-m}\sigma_{-n}$, we have from this

Lemma 4. *Let α and β be 3-braids. If $Q_\alpha(x) = Q_\beta(x)$, then* $Q_{\alpha(m,n)}(x) = Q_{\alpha(-m,-n)}(x) = Q_{\beta(m,n)}(x) = Q_{\beta(-m,-n)}(x)$.

Remark 1. We can also prove the following by induction:

$$(Q_\alpha(x) + \mu)(Q_{\alpha(m,n)}(x) + Q_{\alpha(m+n)}(x)) \\ = (Q_{\alpha(m)}(x) + Q_\alpha(x))(Q_{\alpha(n)}(x) + Q_\alpha(x)).$$

Remark 2. Let $\alpha = S_2^2 S_1^2$. Then the 2-bridge knots and links 10_{14}, 10_{31}, 10_{19}, 10_{36}, 9_4^2, 9_{10}^2 given above have diagrams $G_{\alpha(-2,1)}$, $G_{\alpha(2,-1)}$, $G_{\alpha(2,1)}$, $G_{\alpha(-2,-1)}$, $G_{\alpha(1,1)}$, $G_{\alpha(-1,-1)}$!, respectively.

Suppose that α is a pure 3-braid and m, n are even integers. Let $\nabla_\alpha(z)$ and $\nabla_{\alpha(m,n)}(z)$ be the Conway polynomials of the 2-bridge knots with diagrams G_α and $G_{\alpha(m,n)}$, respectively. (See [3]. Substituting $z = t^{1/2} - t^{-1/2}$, we obtain the Alexander polynomials.) Then we can prove the following by induction:

Lemma 5. $\nabla_{\alpha(m,n)}(z) = \frac{mn}{4}\nabla_\alpha(z)^3 + \nabla_\alpha(z)$.

Suppose that α is a pure 3-braid and m, n are even integers. Let $\nabla_\alpha(t_1, t_2)$ and $\nabla_{\alpha(m,n)}(t_1, t_2)$ be the Conway potential function of the 2-bridge links with diagrams H_α and $H_{\alpha S_1^m \alpha^{-1} S_1^n \alpha}$, respectively, where the links are oriented so that they coincide if $mn = 0$. (See [3]. Substituting t_i for $t_i^{1/2}$, $i = 1, 2$, we obtain the 2-variable Alexander polynomial.) Then we can prove the following by induction:

Lemma 6.

$$\nabla_{\alpha(m,n)}(t_1, t_2) = \frac{mn}{4}(t_1 - t_1^{-1})^2(t_2 - t_2^{-1})\nabla_\alpha(t_1, t_2)^3 + \nabla_\alpha(t_1, t_2).$$

2. Proof of theorem

Knot Case. Let $\alpha = S_2^2 S_1^4$ and $\varepsilon_i u_i$ be either $(2\varepsilon_i, 4\varepsilon_i)$ or $(4\varepsilon_i, 2\varepsilon_i)$, $\varepsilon_i = \pm 1$. We define a pure 3-braid $\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]$ by

$$\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_i u_i, \varepsilon_{i+1} u_{i+1}] = (\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_i u_i])(\varepsilon_{i+1} u_{i+1}),$$

where we interpret $\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]$ as α if $p = 0$. Let $K_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$ be the oriented 2-bridge knot with diagram $G_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$, so K_α is 5_2 in the table of [15]. Let $\mathscr{K}_{\alpha;\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p}$ be the set of 2^p knots $K_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$, $\varepsilon_i u_i = (2\varepsilon_i, 4\varepsilon_i)$ or $(4\varepsilon_i, 2\varepsilon_i)$, and $\mathscr{K}_{\alpha,p}$ be the union $\bigcup_{\varepsilon_i = \pm 1} \mathscr{K}_{\alpha;\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p}$, so $\mathscr{K}_{\alpha,0} = \{K_\alpha\}$. Then all the knots in $\mathscr{K}_{\alpha;\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p}$ are skein-equivalent by [8, Propositions 3.1 and 3.2],, and mutually nonisotopic by [8, Lemma 3.1]. All the knots in $\mathscr{K}_{\alpha,p}$ share the same Q and Alexander polynomials by Lemmas 4 and 5. Since for a 2-bridge knot the minimal crossing number equals the maximal degree of the Q polynomial plus one [11, 13], they have the same minimal crossing number, which we denote by $c_{\alpha,p}$, so $c_{\alpha,0} = 5$. We denote the Jones polynomial of $K_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$ (resp. K_α) by $V_{\alpha;\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p}(t)$ (resp. $V_\alpha(t) = t - t^2 + 2t^3 - t^4 + t^5 - t^6$) and the writhe of $G_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$ (resp. G_α) by $w_{\alpha;\varepsilon_1,\varepsilon_2,\dots,\varepsilon}$ (resp. $w_\alpha = -6$). From Lemma 1 (ii), we have

$$\begin{aligned}&\langle G_{\alpha[\varepsilon_1 u_1, \dots, \varepsilon_p u_p, \varepsilon u]}\rangle \\ &\quad = A^{6\varepsilon}\langle G_{\alpha[\varepsilon_1, \dots, \varepsilon_p]}\rangle\{A^{-16\varepsilon} + A^{-8\varepsilon} - 1 + (A^{-16\varepsilon} - 1)(A^{-8\varepsilon} - 1)d^{-2} \\ &\qquad \times \langle G_{\alpha[\varepsilon_1 u_1, \dots, \varepsilon_p u_p]}\rangle\langle G_{\alpha[\varepsilon_1 u_1, \dots, \varepsilon_p u_p]}!\rangle\}.\end{aligned}$$

Because $V_{\alpha;\varepsilon_1,\dots,\varepsilon_p}(A^4) = (-A^3)^{-w_{\alpha;\varepsilon_1,\dots,\varepsilon_p}}\langle G_{\alpha[\varepsilon_1 u_1, \dots, \varepsilon_p u_p]}\rangle$ and $w_{\alpha;\varepsilon_1,\dots,\varepsilon_p} = w_\alpha - 4(\varepsilon_1 + \cdots + \varepsilon_p)$, we have

$$\begin{aligned}V_{\alpha;\varepsilon_1,\dots,\varepsilon_p,\varepsilon}(t) = t^\varepsilon V_{\alpha;\varepsilon_1,\dots,\varepsilon_p}(t)\{t^\varepsilon + t^{3\varepsilon} - t^{5\varepsilon} \\ + (t^\varepsilon - 1)^2(t^{2\varepsilon} + 1)V_{\alpha;\varepsilon_1,\dots,\varepsilon_p}(t)V_{\alpha;\varepsilon_1,\dots,\varepsilon_p}(t^{-1})\}.\end{aligned}$$

Let $R_{\alpha;\varepsilon_1,\dots,\varepsilon_p}$ and $r_{\alpha;\varepsilon_1,\dots,\varepsilon_p}$ be the maximal and minimal degrees of $V_{\alpha;\varepsilon_1,\dots,\varepsilon_p}(t)$. Then $c_{\alpha,p} = R_{\alpha;\varepsilon_1,\dots,\varepsilon_p} - r_{\alpha;\varepsilon_1,\dots,\varepsilon_p}$ [10, 14, 16]. It is easy to see that

$$\begin{aligned}R_{\alpha;\varepsilon_1,\dots,\varepsilon_p,\varepsilon} &= R_{\alpha;\varepsilon_1,\dots,\varepsilon_p} + c_{\alpha,p} + 3\varepsilon + 2, \text{ and} \\ r_{\alpha;\varepsilon_1,\dots,\varepsilon_p,\varepsilon} &= R_{\alpha;\varepsilon_1,\dots,\varepsilon_p} - 2c_{\alpha,p} + 3\varepsilon - 2.\end{aligned}$$

Thus we have $c_{\alpha,p+1} = 3c_{\alpha,p} + 4 = 3^{p+1}(c_{\alpha,0} + 2) - 2 = 7 \cdot 3^{p+1} - 2$, and $R_{\alpha;\varepsilon_1,\dots,\varepsilon_p} = R_\alpha + (c_{\alpha,0} + 2)(3^p - 1)/2 + 3(\varepsilon_1 + \cdots + \varepsilon_p) = (7 \cdot 3^p + 5)/2 + 3(\varepsilon_1 + \cdots + \varepsilon_p)$, which shows that at least $p + 1$ polynomials in $V_{\alpha;\varepsilon_1,\dots,\varepsilon_p}(t)$'s are distinct.

Remark 3. $\mathscr{K}_{\alpha,\varepsilon_1} = \{K_{\alpha(2\varepsilon_1,4\varepsilon_1)}, K_{\alpha(4\varepsilon_1,2\varepsilon_1)}\}$, $c_{\alpha,1} = 19$, $R_{\alpha;\varepsilon_1} = 3\varepsilon_1 + 13$, $r_{\alpha;\varepsilon_1} = 3\varepsilon_1 - 6$.

$$\begin{aligned}\mathscr{K}_{\alpha;\varepsilon_1,\varepsilon_2} = \{&K_{\alpha(2\varepsilon_1,4\varepsilon_1)(2\varepsilon_2,4\varepsilon_2)}, K_{\alpha(2\varepsilon_1,4\varepsilon_1)(4\varepsilon_2,2\varepsilon_2)}, \\ &K_{\alpha(4\varepsilon_1,2\varepsilon_1)(2\varepsilon_2,4\varepsilon_2)}, K_{\alpha(4\varepsilon_1,2\varepsilon_1)(4\varepsilon_2,2\varepsilon_2)}\},\end{aligned}$$

$c_{\alpha,2} = 65$, $R_{\alpha;\varepsilon_1,\varepsilon_2} = 34 + 3(\varepsilon_1 + \varepsilon_2)$. We can see by a computer calculation that $V_{\alpha;1,-1}(t) \neq V_{\alpha;-1,1}(t)$.

Link case. We change the definition of the pure 3-braid $\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]$ as follows:

$$\alpha = S_2^2 S_1^2 S_2^{-2};\ \alpha[\varepsilon_1 u_1, \dots, \varepsilon_i u_i, \varepsilon_{i+1} u_{i+1}] = \beta S_1^{2m} \beta^{-1} S_1^{2n} \beta,$$

where $\beta = [\varepsilon_1 u_1, \dots, \varepsilon_i u_i]$ and $\varepsilon_{i+1} u_{i+1} = (2m, 2n)$. Consider the oriented 2-bridge link $L_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$ having diagram $H_{\alpha[\varepsilon_1 u_1, \varepsilon_2 u_2, \dots, \varepsilon_p u_p]}$. Then we can prove this case in the same way as the other, using lemmas similar to Lemmas 1–4 and Lemma 6.

Remark 4. If $\varepsilon_i u_i = (2\varepsilon_i, 2\varepsilon_i)$ and $\alpha = S_2^{a_1} S_1^{b_1} \dots S_2^{a_n} S_1^{b_n}$, a_i, $b_i = \pm 2$, then each set $\mathscr{K}_{\alpha;\varepsilon_1,\dots,\varepsilon_p}$ consists of a single fibered 2-bridge knot. See, for example, [6, Lemma 6.3]. Thus we can prove: There exist arbitrarily many fibered 2-bridge knots (resp. links) with the same Q and Alexander polynomials but mutually distinct Jones polynomials.

References

1. R. D. Brandt, W. B. R. Lickorish, and K. C. Millett, *A polynomial invariant for unoriented knots and links*, Invent. Math. **84** (1986), 563–573.
2. G. Burde and H. Zieschang, *Knots*, De Gruyter Studies in Math., vol. 5, Walter de Gruyter, Berlin and New York, 1985.
3. J. H. Conway, *An enumeration of knots and links*, in Computational Problems in Abstract Algebra (J. Leech, ed.), Pergamon Press 1969, 329–358.
4. C. F. Ho, *A new polynomial invariant for knots and links—preliminary report*, Abstracts Amer. Math. Soc. **6** (1985), 300.
5. V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.
6. T. Kanenobu, *Examples on polynomial invariants of knots and links*, Math. Ann. **275** (1986), 555–572.
7. ——, *Relations between the Jones and Q polynomials for 2-bridge and 3-braid links*, Math. Ann. **285** (1989), 115–124.
8. ——, *Examples on polynomial invariants of knots and links* II, Osaka J. Math. **26** (1989), 465–482.
9. T. Kanenobu and T. Sumi, in preparation.
10. L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.
11. M. E. Kidwell, *On the degree of the Brandt-Lickorish-Millet-Ho polynomial of a link*, Proc. Amer. Math. Soc. **100** (1987), 755–762.
12. W. B. R. Lickorish and K. C. Millett, *A polynomial invariant of oriented links*, Topology **26** (1987), 107–141.
13. T. Miyauchi, *On the highest degree of absolute polynomials of alternating links*, Proc. Japan Acad. Ser. A **63** (1987), 174–177.
14. K. Murasugi, *Jones polynomials of alternating links*, Trans. Amer. Math. Soc. **295** (1986), 147–174.
15. D. Rolfsen, *Knots and links*, Math. Lecture Series no. 7, Publish or Perish, Berkeley, 1976.
16. M. B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology **26** (1987), 297–309.

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