ON THE DIVISOR OF INVOLUTIONS IN AN ELLIPTIC MODULAR SURFACE

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ABSTRACT. Let $E \rightarrow X$ be an elliptic modular surface and S the tangential ruled surface of a projective embedding of X . The divisor that collects the involutions of the elliptic fibers of E is precisely the branch locus of $E \rightarrow S$ (at least generically). In this paper, we present two theorems that characterize this divisor in terms of the action of the group of modular automorphisms. These results extend work of D. Burns [1].

0. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

The purpose of this paper is to present two theorems that together extend a result of D. Burns [1, Theorem 2]. This extension gives a rather complete picture of the invariant-theoretical question addressed there. We believe that these results will prove useful in the construction of elliptic modular surfaces with prescribed finite group as automorphisms.

Thus, suppose that Γ is a torsion-free subgroup of finite index in $\Gamma(1) := SL_2(\mathbb{Z})$, and let $X = X(\Gamma)$ be the usual modular curve attached to Γ [cf. 3 for details on modular curves]. There is an elliptic surface $E = E(\Gamma)$ fibered over X such that, whenever $\Gamma \triangleleft \Delta \leq \Gamma(1)$, the finite group $\mathcal{G} := \Delta/\Gamma$ acts on E in a very nice way [cf. 1 or 4]. In particular, $-1 \in \Gamma(1)$ induces an involution of E , and the quotient $\hat{S} := S/\langle -1 \rangle$ is birationally ruled over X . In [1], Burns constructs intrinsically a minimal model S of $\hat{S}$ (at least for almost all Γ). This *geometric model* S is a ruled surface over X , and Burns shows [1, Theorem 1] that extrinsically S is the tangential ruled surface of any projective embedding of X . Hence, to construct E explicitly from X and $\mathcal{G}$ one should pinpoint the branch locus of the map $E \rightarrow S$.

One component of this branch locus is the zero-section $Z \subset S$. A second component is the image ϑ of a smooth 3-fold section $\vartheta_E \subset E$. The divisor class of ϑ is $D := 3Z + 3K_X + \kappa_\Gamma$, where a canonical divisor K_X and the

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divisor of cusps κ_Γ are pulled back from X to S . Thus, ϑ lies in the space $|D|^\mathcal{G}$ of $\mathcal{G}$ -invariant effective divisors linearly equivalent to D . Now, $|D|^\mathcal{G}$ is equal to $\bigcup \mathbf{P}V_\lambda$, where $\{V_\lambda\}$ are the eigenspaces of $H^0(S, \mathcal{O}_S(D))$, which afford linear characters $\{\lambda\}$ of $\mathcal{G}$. In this context Burns' result is as follows:

(I) **Theorem** [1, Theorem 2]. *If $\Delta = \Gamma(1)$ then $\dim H^0(S, \mathcal{O}_S(D))^\mathcal{G} = 1$. In particular, in this case, if $\mathcal{G} = \mathcal{D}^1(\mathcal{G})$, then ϑ is the unique $\mathcal{G}$ -invariant, effective divisor in its divisor class.*

The two theorems that complete this picture are these.

(II) **Theorem.** *Suppose $\Delta \leq \Gamma(1)$ has finite index (but is otherwise arbitrary), and let Y be the modular curve attached to Δ . Then there is a rank-4 vector bundle Ξ_Δ over Δ such that the following hold :*

- (a) $h^0(Y, \Xi_\Delta) = 1 + \frac{1}{4}(3I(\Delta) - n_2(\Delta) - 2n_\infty(\Delta))$;
- (b) *for any $\Gamma \triangleleft \Delta$ that is torsion free, of finite index, and nondegenerate at ∞ , we have a filtration of length 4 on $H^0(S, \mathcal{O}_S(D))^\mathcal{G}$, such that $\text{Gr}(H^0(S, \mathcal{O}_S(D))^\mathcal{G}) \cong H^0(Y, \Xi_\Delta)$.*

(III) **Theorem.** *Suppose in addition that $\Delta \triangleleft \Gamma(1)$, and set $\mathcal{H} = \Gamma(1)/\Delta$. Then $\dim H^0(Y, \Xi_\Delta)^\mathcal{H} = 1$.*

Note that (II) implies (I) directly, since $I(\Delta) = n_2(\Delta) = n_\infty(\Delta) = 1$ when $\Delta = \Gamma(1)$. In fact, the proof of (II) is merely a careful adaptation of Burns' proof of (I).

Specifically, the isomorphism (II-b) holds for the generic case where all the cusps of X are ramified over their images under the natural map $X \rightarrow Y$ (cf. hypothesis (∞) in §1).

The vector bundle Ξ_Δ is in fact a trivial extension of line bundles, and its definition is given in §1, where a complete list of definitions, notation, and preliminary results is collected. As yet we have no more natural definition of Ξ_Δ than is given there.

Finally, theorem (III) is somewhat remarkable in that, in general, there is no action of $\mathcal{H}$ on E , S , or X . However the action of $\mathcal{H}$ in $|D|^\mathcal{G}$ picks out ϑ uniquely, without reference to Γ . This suggests that an explicit construction of ϑ can be done rather uniformly, although we have as yet to see how this is done.

1. DEFINITIONS, NOTATION, AND PRELIMINARY RESULTS

We begin by collecting most of the definitions and notation in use throughout the paper.

$\Gamma(1) = SL_2(\mathbf{Z})$, and Δ is a subgroup of finite index;

$Y = \mathbf{H}/\Delta \cup \text{cusps } \Delta$ is the modular curve (Riemann surface) attached to Δ ; its genus is written $p_g(\Delta)$ or $p_g(Y)$; here $\mathbf{H} \subset \mathbf{C}$ is the upper half-plane;

$n_\infty(\Delta) = \# \text{cusps} \Delta = \#\{P \in Y \mid |\Delta_z| = \infty\}$; Δ_z will denote the point-stabilizer (isotropy subgroup) of z in Δ ; here $z \in \mathbf{H}\mathbf{P}^1(\mathbf{Q})$ represents P ;

$n_2(\Delta) = \#\{\text{elliptic points of order 2 for } \Delta\} = \#\{P \in Y \mid |\Delta_z| = 4\}$;

$n_3(\Delta) = \#\{\text{elliptic points of order 3 for } \Delta\} = \#\{P \in Y \mid |\Delta_z| = 3 \text{ or } 6\}$;

$I(\Delta) = [\Gamma(1) : \{\pm 1\} \cdot \Delta] = \text{group index of } \{\pm 1\} \cdot \Delta \text{ in } \Gamma(1)$.

The Riemann–Hurwitz relation, applied to the map $Y \rightarrow X(\Gamma(1)) \cong \mathbf{P}^1$, yields the following:

(R–H) **Lemma.** $p_g(\Delta) = 1 + \frac{1}{12}I(\Delta) - \frac{1}{4}n_2(\Delta) - \frac{1}{3}n_3(\Delta) - \frac{1}{2}n_\infty(\Delta)$.

We continue with the definitions and notation:

$K_\Delta = K_Y$ is a canonical divisor for Y ;

κ_Δ is the divisor of cusps, each counted with multiplicity 1;

ε_Δ^k is the divisor of elliptic points of order k , each counted with multiplicity 1; and $\varepsilon_\Delta = \varepsilon_\Delta^2 + \varepsilon_\Delta^3$;

Ω_Y^1 is the canonical sheaf or sheaf of holomorphic 1-differentials on Y ; $\Omega_Y^2 = \Omega_Y^1 \otimes \Omega_Y^1$, etc;

Θ_Y is the (holomorphic) tangent sheaf on Y , isomorphic to the dual of Ω_Y^1 ; $\Theta_Y^2 = \Theta_Y \otimes \Theta_Y$, etc

Suppose that Γ is a *torsion-free* normal subgroup of finite index in Δ . Thus, $n_2(\Gamma) = n_3(\Gamma) = 0$, and $-1 \notin \Gamma$. Let $X = X(\Gamma) = \mathbf{H}/\Gamma \cup \text{cusps } \Gamma$. We have a natural action of $\mathcal{G} := \Delta/\Gamma$ on X , and Y is canonically isomorphic with $X/\mathcal{G}$. We will use the notation $p_g(X) = p_g(\Gamma)$, $n_\infty(\Gamma)$, $I(\Gamma) = (1/2)[\Gamma(1) : \Delta] \cdot |\mathcal{G}|$, $K_X = K_\Gamma$, κ_Γ , Ω_X^1 , Θ_X , etc. exactly as above. Also, if P is a cusp (for Δ or Γ) then its *width* b_P is the ramification index at P for the natural map to $X(\Gamma(1)) \cong \mathbf{P}^1$, as above. We will assume that the inclusion $\Gamma \triangleleft \Delta$ is nondegenerate at ∞ , in the following sense:

(∞) *Hypothesis.* Each cusp for Γ is ramified over its image under $X \rightarrow Y$. That is, if $P \in \text{cusps } \Gamma$ and Q is its image in Y , then the integer b_P/b_Q is at least 2.

As a consequence, the width of every cusp for Γ is at least 2. Note that as the conjugacy class of the cusp element $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in $\Gamma(1)$ generates all of $\Gamma(1)$, we are always assured of (∞) when $\Delta = \Gamma(1)$.

As noted in the introduction, there is an elliptic fibering $E \rightarrow X$ and zero-section $Z_E \subset E$ that admit $\mathcal{N}_{\Gamma(1)}(\Gamma)/\Gamma$ as automorphisms. The involution induced by $-1 \in \Gamma(1)$ inverts each elliptic fiber E_P , $P \in X \setminus \text{cusps } \Gamma$. There are two components of $\text{Fix}_E(-1)$: Z_E and a smooth 3-fold section ϑ_E that collects the involutions of the elliptic fibers. The quotient $E/\langle -1 \rangle$ is smooth and birationally ruled over X . Hypothesis (∞) implies that each cusp width is at least 2, so that there is a *geometric model* S of $E/\langle -1 \rangle$ [cf. 1, pp. 6ff.].

Let Z be the zero-section of S and ϑ the image of ϑ_E in S . If $\mathcal{N}_Z$ denotes the normal sheaf to Z in S , then from [1, Theorem 1] we have that $\mathcal{N}_Z \cong \Theta_X$ on $Z \cong X$. Now ϑ is in the divisor class of $D := 3Z + 3K_X + \kappa_\Gamma$, where K_X and κ_Γ are pulled back to S .

Two final pieces of notation: if A is a space, B is a divisor, and $\mathcal{L}$ is an $\mathcal{O}_A$ -module, then $\mathcal{L}(B)$ will mean the product of $\mathcal{L}$ with the line bundle (invertible sheaf) $\mathcal{O}_A(B)$. If $\mathcal{F}$ is a group, the functor $( )^{\mathcal{F}}$ will mean the fixed points of $\mathcal{F}$. We now proceed with some of the preliminary results that will be needed in the proofs of (II) and (III).

We recall Maschke's Theorem:

(M) **Lemma.** *When $\mathcal{F}$ is finite, the functor $( )^{\mathcal{F}}$ is exact in characteristic 0.*

The following is a consequence of Grauert's Theorem [cf. 2, V.2.4, p. 371].

(G) **Lemma.** *Let A be a divisor on a ruled surface $R \xrightarrow{P} C$, with $A \cdot F \geq 0$ F is where a fiber. Then $H^i(R, \mathcal{O}_R(A)) \cong H^i(C, p_* \mathcal{O}_R(A))$ for all $i \geq 0$.*

The following is well known. I include a proof for the sake of completeness:

(Q) **Lemma.** *Suppose B is the quotient of a smooth curve A by the finite group $\mathcal{F}$. Let P be any point of A , Q its image in B , and e_Q the ramification index. Then any $\mathcal{F}$ -invariant differential α of A descends to a differential β of B with:*

$$\text{ord}_Q(\beta) = \frac{\text{ord}_P(\alpha) + \deg(\alpha)}{e_Q} - \deg(\alpha).$$

This formula holds whether $\deg(\alpha)$ is positive, negative, or 0. Moreover, any differential β of B pulls back to an $\mathcal{F}$ -invariant differential α of A , and the above equation remains valid.

Proof. We may choose local coordinates z, w around P, Q , respectively, so that locally the mapping $A \rightarrow B$ looks like $z \mapsto w = z^e$, where $e = e_Q = |\mathcal{F}_P|$. In this setup, $\mathcal{F}_P$ acts via multiplication by $\zeta_e = \exp(2\pi i/e)$. If α near P is $(a_N z^N + \dots) dz^k$, then substitution of $\zeta_e z$ leaves α invariant if and only if $a_t = 0$ for $t \not\equiv -k \pmod{e}$. Thus α must have the form

$$(a_{-k+ne} z^{-k+ne} + \dots) dz^k = e^{-k} (a_{-k+ne} w^{n-k} + \dots) dw^k,$$

and the claim follows. The argument is similar for $\alpha = (a_N z^N + \dots) \left(\frac{d}{dz}\right)^k$. $\square$

The vector bundle Ξ_Δ of (II) is defined as $\Xi_\Delta = \mathcal{O}_Y \oplus \Xi'_\Delta$, where $\Xi'_\Delta = \Omega_Y^1(\kappa_\Delta) \oplus \Omega_Y^2(\kappa_\Delta + \varepsilon_\Delta) \oplus \Omega_Y^3(2\kappa_\Delta + \varepsilon_\Delta^2 + 2\varepsilon_\Delta^3)$. We will need the following

estimates on the Serre duals of components of Ξ'_Δ :

(E₁) **Lemma.** $\deg(\Theta_Y(-\kappa_\Delta - \varepsilon_\Delta)) < 0$.

Proof. $\deg(\Theta_Y(-\kappa_\Delta - \varepsilon_\Delta)) = 2 - 2p_g(\Delta) - n_\infty(\Delta) - n_2(\Delta) - n_3(\Delta) < 0$, unless perhaps $p_g(\Delta) = 0$ and

$$\begin{cases} n_\infty(\Delta) + n_2(\Delta) + n_3(\Delta) \leq 2 \\ 6n_\infty(\Delta) + 3n_2(\Delta) + 4n_3(\Delta) = 12 + I(\Delta) \quad (\text{from (R-H)}), \end{cases}$$

an impossibility, since $I(\Delta) \geq 1$. $\square$

(E₂) **Lemma.** $\deg(\Theta_Y^2(-2\kappa_\Delta - \varepsilon_\Delta^2 - 2\varepsilon_\Delta^3)) < 0$.

Proof. $\deg(\Theta_Y^2(-2\kappa_\Delta - \varepsilon_\Delta^2 - 2\varepsilon_\Delta^3)) = 4 - 4p_g(\Delta) - 2n_\infty(\Delta) - n_2(\Delta) - 2n_3(\Delta) < 0$, unless perhaps $p_g(\Delta) = 0$ and:

$$\begin{cases} 2n_\infty(\Delta) + n_2(\Delta) + 2n_3(\Delta) \leq 4 \\ 6n_\infty(\Delta) + 3n_2(\Delta) + 4n_3(\Delta) = 12 + I(\Delta). \end{cases}$$

Now $n_\infty(\Delta) \geq 1$ and $I(\Delta) \geq 1$ for any $\Delta \leq \Gamma(1)$. We are thus forced to $n_\infty(\Delta) = 1$, and thence to an impossibility. $\square$

Finally, we will refer to the Serre Duality Theorem as (SD).

2. PROOFS OF THE MAIN THEOREMS

The proofs of the main theorems proceed in several steps. We begin with an analysis of three fundamental sequences of [1, p. 15].

$$\begin{array}{ccccccc} (1) & 0 & \rightarrow & \mathcal{O}_S(D - Z) & \rightarrow & \mathcal{O}_S(D) & \rightarrow \mathcal{O}_X(\kappa_\Gamma) \rightarrow 0 \\ (2) & 0 & \rightarrow & \mathcal{O}_S(D - 2Z) & \rightarrow & \mathcal{O}_S(D - Z) & \rightarrow \Omega_X^1(\kappa_\Gamma) \rightarrow 0 \\ (3) & 0 & \rightarrow & \mathcal{O}_S(3K_X + \kappa_\Gamma) & \rightarrow & \mathcal{O}_S(D - 2Z) & \rightarrow \Omega_X^2(\kappa_\Gamma) \rightarrow 0 \\ & & & \parallel & & & \\ & & & \mathcal{O}_S(D - 3Z) & & & \end{array}$$

For the next three lemmas, we apply the functor $H^*( )^{\mathcal{E}}$ to these three sequences.

$(H^*(3))^{\mathcal{F}}$ **Lemma.** *We have that*

$$\dim H^0(S, \mathcal{O}_S(D - 2Z))^{\mathcal{F}} = h^0\left(Y, \Omega_Y^2(\kappa_{\Delta} + \varepsilon_{\Delta}) \oplus \Omega_Y^3(2\kappa_{\Delta} + \varepsilon_{\Delta}^2 + 2\varepsilon_{\Delta}^3)\right)$$

and

$$H^1(S, \mathcal{O}_S(D - 2Z))^{\mathcal{F}} = 0.$$

Proof. We use the standard symbol h^i to mean $\dim H^i$. From (3) we obtain:

$$\begin{array}{ccc} H^0\left(Y, \Omega_Y^3(2\kappa_{\Delta} + \varepsilon_{\Delta}^2 + 2\varepsilon_{\Delta}^3)\right) & & 0 \\ \parallel \wr (Q) & & \downarrow \\ H^0(X, \Omega_X^3(\kappa_{\Gamma}))^{\mathcal{F}} & \stackrel{(G)}{\cong} & H^0(S, \mathcal{O}_S(3K_X + \kappa_{\Gamma}))^{\mathcal{F}} \\ & & \downarrow \\ & & H^0(S, \mathcal{O}_S(D - 2Z))^{\mathcal{F}} \\ & & \downarrow \\ H^0(Y, \Omega_Y^2(\kappa_{\Delta} + \varepsilon_{\Delta})) & \stackrel{(Q)}{\cong} & H^0(X, \Omega_X^2(\kappa_{\Gamma}))^{\mathcal{F}} \\ & & \downarrow \\ H^1(X, \Omega_X^3(\kappa_{\Gamma}))^{\mathcal{F}} & \stackrel{(G)}{\cong} & H^1(S, \mathcal{O}_S(3K_X + \kappa_{\Gamma}))^{\mathcal{F}} \\ \parallel \wr (SD) & & \downarrow \\ H^0(X, \Theta_X^2(-\kappa_{\Gamma}))^{\mathcal{F}} & & H^1(S, \mathcal{O}_S(D - 2Z))^{\mathcal{F}} \\ \parallel \wr (Q) & & \downarrow \\ H^0\left(Y, \Theta_Y^2(-2\kappa_{\Delta} - \varepsilon_{\Delta}^2 - 2\varepsilon_{\Delta}^3)\right) & & H^1(X, \Omega_X^2(\kappa_{\Gamma}))^{\mathcal{F}} \\ \parallel (E_2) & & \parallel \wr (SD) \\ 0 & & H^0(X, \Theta_X(-\kappa_{\Gamma}))^{\mathcal{F}} \\ & & \parallel \wr (Q) \\ & & H^0(Y, \Theta_Y(-\kappa_{\Delta} - \varepsilon_{\Delta})) \\ & & \parallel (E_1) \\ & & 0 \end{array}$$

This gives the lemma. $\square$

$(H^*(2))^{\mathcal{F}}$ **Lemma.** *We have that $\dim H^0(S, \mathcal{O}_S(D - Z))^{\mathcal{F}} = h^0(Y, \Xi'_{\Delta})$ and $H^1(S, \mathcal{O}_S(D - Z))^{\mathcal{F}} = 0$.*

Proof. We now consider the cohomology of (2):

$$\begin{array}{c}
 0 \\
 \downarrow \\
 H^0(S, \mathcal{O}_S(D - 2Z))^{\mathcal{E}} \\
 \downarrow \\
 H^0(S, \mathcal{O}_S(D - Z))^{\mathcal{E}} \\
 \downarrow \\
 H^0(Y, \Omega_Y^1(\kappa_\Delta)) \stackrel{(Q)}{\cong} H^0(X, \Omega_X^1(\kappa_\Gamma))^{\mathcal{E}} \\
 \downarrow \\
 0 = H^1(S, \mathcal{O}_S(D - 2Z))^{\mathcal{E}} \\
 \downarrow \\
 H^1(S, \mathcal{O}_S(D - Z))^{\mathcal{E}} \\
 \downarrow \\
 0 = H^0(X, \mathcal{O}_X(-\kappa_\Gamma))^{\mathcal{E}} \stackrel{(SD)}{\cong} H^1(X, \Omega_X^1(\kappa_\Gamma))^{\mathcal{E}}
 \end{array}$$

Lemma $(H^*(3))^{\mathcal{E}}$ finishes the present lemma. $\square$

$(H^*(1))^{\mathcal{E}}$ **Lemma.** We have $\dim H^0(S, \mathcal{O}_S(D))^{\mathcal{E}} = h^0(Y, \Xi_\Delta)$.

Proof. Finally, the cohomology of (1) yields:

$$\begin{array}{c}
 0 \\
 \downarrow \\
 H^0(S, \mathcal{O}_S(D - Z))^{\mathcal{E}} \\
 \downarrow \\
 H^0(S, \mathcal{O}_S(D))^{\mathcal{E}} \\
 \downarrow \\
 H^0(Y, \mathcal{O}_Y) \stackrel{(Q)}{\cong} H^0(X, \mathcal{O}_X(\kappa_\Gamma))^{\mathcal{E}} \\
 \downarrow \\
 0 = H^1(S, \mathcal{O}_S(D - Z))^{\mathcal{E}}
 \end{array}$$

Now apply $(H^*(2))^{\mathcal{E}}$. $\square$

Proof of Theorem (II). The isomorphism of $Gr(H^0(S, \mathcal{O}_S(D))^{\mathcal{E}})$ with $H^0(Y, \Xi_\Delta)$ was established above. It remains to calculate the dimension.

SD and the estimates (E_1) and (E_2) combine to give that $H^1(Y, \Xi'_\Delta) = 0$. An application of the Riemann–Roch Theorem to the components of Ξ'_Δ yields

$$\begin{aligned}
 h^0(Y, \Xi_\Delta) &= 1 + 3 - 3p_g(\Delta) \\
 &\quad + 12p_g(\Delta) - 12 + 4n_\infty(\Delta) + 2n_2(\Delta) + 3n_3(\Delta) \\
 &= 1 + \frac{1}{4}(3I(\Delta) - n_2(\Delta) - 2n_\infty(\Delta)),
 \end{aligned}$$

as claimed. $\square$

Remark. It is curious that the 3-elements of Δ have no bearing on the dimension of $H^0(Y, \Xi_\Delta)$, even though they figure crucially in the definition of Ξ_Δ given here. Perhaps a more natural definition of Ξ_Δ is possible.

Examples. (i) The commutator subgroup $\Delta = \mathcal{D}^1(\Gamma(1))$ is a free group on 2 generators. We have $I(\Delta) = 6$ and $n_\infty(\Delta) = 1$. Thus $h^0(Y, \Xi_\Delta) = 5$ and $p_g(\Delta) = 1$.

(ii) Consider

$$\Delta = \Gamma(N) := \left\{ m \in \Gamma(1) \mid m \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{N} \right\}.$$

We have

$$I(\Delta) = \begin{cases} 6 & N = 2 \\ \frac{1}{2}N\varphi(N)\psi(N) & N > 2 \end{cases},$$

and

$$n_\infty(\Delta) = \begin{cases} 3 & N = 2 \\ \frac{1}{2}\varphi(N)\psi(N) & N > 2, \end{cases}$$

where $\varphi(N) = N \prod_{p|N} (1 - (1/p))$ and $\psi(N) = N \prod_{p|N} (1 + (1/p))$ are the standard totients. Moreover, Δ is free on $2p_g(\Delta) + n_\infty(\Delta) - 1 = 1 + (1/6)I(\Delta)$ generators when $N > 2$, while $\Gamma(2)$ is the direct product of $\{\pm 1\}$ and a free group on 2 generators. This gives:

$$h^0(Y, \Xi_\Delta) = \begin{cases} 4 & N = 2 \\ 1 + \frac{1}{8}(3N - 2)\varphi(N)\psi(N) & N > 2. \end{cases}$$

Proof of Theorem (III). As noted above, if any cusp for Δ is *not* ramified over ∞ , then $\Delta = \Gamma(1)$, in which case $h^0(Y, \Xi_\Delta) = 1$. We assume that $\Delta \neq \Gamma(1)$.

Over the elliptic points i and $\rho = \frac{1}{2}(-1 + \sqrt{3}i)$, the ramification indices are:

$$\begin{aligned} e_i &= 1 \text{ resp. } 2 & \text{ as } \varepsilon_\Delta^2 \neq 0 \text{ resp. } \varepsilon_\Delta^2 = 0 \\ e_\rho &= 1 \text{ resp. } 3 & \text{ as } \varepsilon_\Delta^3 \neq 0 \text{ resp. } \varepsilon_\Delta^3 = 0. \end{aligned}$$

This, together with lemma (Q), yields:

$$\begin{aligned} H^0(Y, \mathcal{O}_Y)^{\mathcal{X}} &\cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) \cong \mathbf{C}, \\ H^0(Y, \Omega_Y^1(\kappa_\Delta))^{\mathcal{X}} &\cong H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(1)) = 0, \\ H^0(Y, \Omega_Y^2(\kappa_\Delta + \varepsilon_\Delta))^{\mathcal{X}} &\cong H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^2(3)) = 0, \text{ and} \\ H^0\left(Y, \Omega_Y^3\left(2\kappa_\Delta + \varepsilon_\Delta^2 + 2\varepsilon_\Delta^3\right)\right)^{\mathcal{X}} &\cong H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^3(5)) = 0. \end{aligned}$$

This gives the claim. $\square$

3. AN OBSTRUCTION TO A FURTHER EXTENSION OF THEOREM (I)

We give an example to show that, without further hypotheses on $\mathcal{G}$, there may be more than one $\mathcal{G}$ -invariant effective divisor linearly equivalent to ϑ ,

even with $\Gamma \triangleleft \Gamma(1)$. Indeed, consider $\Gamma := \mathcal{D}^1(\Gamma(1))$. Since $\Gamma(1)$ is an amalgamated product $\mathbf{Z}/6 *_{\mathbf{Z}/2} \mathbf{Z}/4$, we see that $\mathcal{G} \cong \mathbf{Z}/12$. Moreover, Γ is free on two generators, with exactly one cusp.

Recall that, for any Γ , S is the projectivization of $\mathcal{E} := p_* \mathcal{O}_S(Z)$. Now $\mathcal{E}$ is an extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \Theta_X \rightarrow 0.$$

As a consequence, $\mathcal{E}$ is normalized, and Z is a distinguished section, whenever X is not rational. Thus, returning to the case $\Gamma = \mathcal{D}^1(\Gamma(1))$, we see that $p_g(\Gamma) = -p_a(S) = 1$, $K_X = 0$, $Z^2 = \deg \mathcal{N}_Z = \deg \Theta_X = 0$, and $K_S = -2Z$ [cf. 2, §V.2].

Since $\mathcal{E}$ is Abelian, $H^0(S, \mathcal{O}_S(\vartheta))$ is a sum of 1-dimensional eigenspaces for $\mathcal{E}$. As noted in the introduction, an eigenvector $f \in H^0(S, \mathcal{O}_S(\vartheta))$ gives a $\mathcal{E}$ -invariant, effective divisor $\vartheta + (f) \in |\vartheta|$, and conversely. Now $h^0(S, \mathcal{O}_S(\vartheta)) = 4$, as one can see from the same calculations that were used in §2. At any rate, Lemma (G) implies that $h^2(S, \mathcal{O}_S(\vartheta)) = 0$, whence:

$$\begin{aligned} \#(|\vartheta|^{\mathcal{E}}) &\geq h^0(S, \mathcal{O}_S(\vartheta)) \\ &\geq \chi(S, \mathcal{O}_S(\vartheta)) \\ &= 1 - 1 + \frac{1}{2}(3Z + \kappa_\Gamma) \cdot (5Z + \kappa_\Gamma) \quad (\text{Riemann-Roch}) \\ &= 4. \end{aligned}$$

Thus it seems that a general characterization of ϑ must include some of its geometric properties.

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