HOMOLOGICAL DIMENSIONS IN A MORITA CONTEXT WITH APPLICATIONS TO SUBIDEALIZERS AND FIXED RINGS

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ABSTRACT. Given a Morita context $(R, S, V, W, \theta, \psi)$, we investigate the relationship between the various homological dimensions of the rings R and S . We then apply these results to two particular examples: subidealizers and fixed rings.

Let R and S be rings, ${}_R V_S, {}_S W_R$ bimodules and $\theta: V \otimes_S W \rightarrow R$ and $\psi: W \otimes_R V \rightarrow S$ bimodule homomorphisms. Consider the array

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}.$$

If θ and ψ satisfy the associativity conditions required to make T a ring, then we call the collection $(R, S, V, W, \theta, \psi)$ a *Morita context* and T the *ring of the Morita context* (see [McCR] for details). Recently, in [C] and [LS], Morita contexts have been used as a tool to relate properties of a ring R and its fixed subring R^G , where G is a group of automorphisms of R . In this paper we also use Morita contexts as a device to obtain information on the homological dimensions of fixed rings and subidealizers.

In the first section, we relate the various homological dimensions of the rings R and T (in a Morita context) under certain assumptions on V or V and W (Lemma 1.2 and Theorem 1.8). If, in addition θ is onto, it is known that T and S are Morita equivalent (see [LS, Lemma 1.2]). Thus we have a relationship between the homological dimensions of R and S .

In §2, we apply these results to two naturally occurring Morita contexts. In the first example we show that if S is a subidealizer of a generative right ideal I of R (i.e., S is a subring of R containing I as a two-sided ideal and $RI = R$), then

$$\text{w.gldim}(R) \leq \text{w.gldim}(S) \leq \max\{\beta + 1, \text{w.gldim}(R)\},$$

where $\beta = \text{w.gldim}(S/I) + \text{wdim}(R/I)_R$ (Theorem 2.1), and $\text{w.gldim}(X)$ denotes the weak global dimension of the ring X . Robson and Small [RS] have

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proved the same result for global dimension. As a corollary we show that if S/I is von Neumann regular, then (Corollary 2.2)

$$\text{w.gldim}(R) \leq \text{w.gldim}(S) \leq \text{w.gldim}(R) + 1.$$

This result improves on [G, Theorem 2.3].

Our second application involves the fixed ring R^G of a ring R , where G is a finite group of automorphisms of R . Recently, Lorenz [L] has obtained a bound on the global dimension of R^G under certain assumptions on R and G , including R flat over R^G . We consider the classical case where the order of G is a unit in R ; then, if R is flat over R^G , we show that

$$\text{r.gldim}(R^G) \leq \text{r.gldim}(R),$$

where $\text{r.gldim}(A)$ denotes the right global dimension of the ring A .

Throughout the first section we will assume that we have a fixed Morita context $(R, S, V, W, \theta, \psi)$ with T the ring of the Morita context. To shorten notation we will use vw (resp. wv) to denote $\theta(v \otimes w)$ (resp. $\psi(w \otimes v)$). Moreover, VW (resp. WV) will be the image of θ (resp. ψ). We let H denote the right ideal $\begin{pmatrix} R & V \\ 0 & 0 \end{pmatrix}$ of T . Clearly H is an $R - T$ bimodule, which is projective as a T -module. Similarly K will be the left ideal $\begin{pmatrix} R & 0 \\ W & 0 \end{pmatrix}$ of T , which is a $T - R$ bimodule and is also projective as a T -module. We note the following useful fact for future reference: $H \otimes_T K \approx R$ as $R - R$ bimodules [LS, Proof of Lemma 2.1].

Given a module M over A , the projective (resp. weak or flat) dimension of M will be denoted by $\text{pdim}(M)_A$ (resp. $\text{wdim}(M)_A$). If there is no chance for confusion the subscript will be omitted. Finally, if M is a module and B a set, then $M^{(B)}$ denotes the direct sum of M indexed by B .

1. DIMENSIONS IN A MORITA CONTEXT

In this section we prove our basic results relating the various homological dimensions of the rings R and T in a general Morita context. This will be done in two steps. First we show that the dimension of R is bounded by the dimension of T if V is flat as a left R -module. Then we obtain a bound for the dimension of T in terms of R and S/WV . The main tools that we will use are the functors ${}_-\otimes_R H$ from $\text{Mod}-R$ to $\text{Mod}-T$ and ${}_-\otimes_T K$ from $\text{Mod}-T$ to $\text{Mod}-R$.

Lemma 1.1. *If M , as a T -module, is the homomorphic image of $H^{(B)}$, for some B , then M projective (respectively flat) over T implies that $M \otimes_T K$ is projective (resp. flat) over R .*

Proof. There exists an exact sequence of T -modules

$$0 \rightarrow A \xrightarrow{g} H^{(B)} \rightarrow M \rightarrow 0.$$

If M is projective, this sequence splits and the projective case follows from the fact that $H \otimes_T K \approx R$. For the flatness part, since H is projective, we

can use Villamayor's characterization of flat modules (see, for example, [R, Theorem 3.57]). First we get the exact sequence

$$0 \rightarrow A \otimes_T K \rightarrow R^{(B)} \rightarrow M \otimes_T K \rightarrow 0.$$

Then let $v = \sum a_i \otimes k_i \in A \otimes_T K$. Since M is flat and $H^{(B)}$ is projective over T , there exists $f \in \text{Hom}_T(H^{(B)}, A)$ such that $f(g(a_i)) = a_i$. Let $h = f \otimes 1_K$, then $h(g \otimes 1_K(v)) = v$. Therefore $M \otimes_T K$ is flat as an R -module.

Lemma 1.2. *If V is flat as a left R -module, then $\text{r.gldim}(R) \leq \text{r.gldim}(T)$ and $\text{w.gldim}(R) \leq \text{w.gldim}(T)$.*

Proof. We prove the global dimension result (the proof for the weak global dimension is similar). Clearly we can assume that $\text{r.gldim}(T) = m < \infty$. Let N be a right R -module. There is an exact sequence of R -modules

$$(*) \quad 0 \rightarrow I \rightarrow R^{(A)} \rightarrow N \rightarrow 0$$

which, since ${}_R V$ is flat, induces the exact sequence of T -modules

$$0 \rightarrow I \otimes_R H \rightarrow H^{(A)} \xrightarrow{\psi} N \otimes_R H \rightarrow 0.$$

From this we deduce that every T -module of the form $X \otimes_R H$, where X is an R -module, is the homomorphic image of $H^{(A)}$, such that the kernel also has the form $Y \otimes_R H$, where Y is an R -module. Thus, for any R -module N , we obtain a long exact sequence of T modules

$$0 \rightarrow P \rightarrow H^{(A_{m-1})} \rightarrow \dots \rightarrow H^{(A_0)} \rightarrow N \otimes_R H \rightarrow 0.$$

Since H is a projective T -module and since $\text{r.gldim}(T) = m$, P must be projective. Also by the above considerations P is a homomorphic image of some $H^{(B)}$. Now applying the exact functor ${}_T K$ to the above sequence and using the fact that $H \otimes_T K \approx R$, we get the sequence of R -modules

$$0 \rightarrow P \otimes_T K \rightarrow R^{(A_{m-1})} \rightarrow \dots \rightarrow R^{(A_0)} \rightarrow N \rightarrow 0.$$

By Lemma 1.1, $P \otimes_T K$ is a projective R -module. Therefore $\text{pdim}(N) \leq m$ and hence the lemma is proved.

We will now work toward getting a bound on the dimensions of the ring T in terms of the dimension of the rings R and S/WV and the dimension of the S -module S/WV .

Lemma 1.3. *Let M be a T -module which is killed by the ideal I . Then*

$$\text{pdim}(M) \leq \text{r.gldim}(T/I) + \text{pdim}(T/I)_T$$

and

$$\text{wdim}(M) \leq \text{w.gldim}(T/I) + \text{wdim}(T/I)_T.$$

Proof. The global dimension part is well known (see, for example, [R, Theorem 9.32]). The weak dimension part has almost the same proof except at the start of the induction (on $\text{wdim}(M)_{T/I}$). To deal with that case, first recall a

result of Lazard which says that a flat module is a direct limit of projective modules (see, for example, [F, Proposition 11.32]). The result then follows from the fact that Tor commutes with direct limits.

The ideal $\begin{pmatrix} R & V \\ W & WV \end{pmatrix}$ of T will be denoted by J . It is not difficult to see that the rings T/J and S/WV are isomorphic.

Lemma 1.4. *Assume that the left S -module W is flat and that $(WV)W = W$, then $\text{pdim}(T/J)_T \leq \text{pdim}(S/WV)_S$ and $\text{wdim}(T/J)_T \leq \text{wdim}(S/WV)_S$.*

Proof. We prove the projective dimension result (the proof for the weak dimension is similar). Again we can assume that $\text{pdim}(S/WV)_S = n < \infty$. Take an S -projective resolution of S/WV

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow S/WV \rightarrow 0.$$

Denote the right ideal $\begin{pmatrix} 0 & 0 \\ W & S \end{pmatrix}$ of T by H' . By our assumption on W , H' is flat as a left S -module. So we get the exact sequence of T -modules

$$0 \rightarrow P_n \otimes_S H' \rightarrow \cdots \rightarrow P_0 \otimes_S H' \rightarrow (S/WV) \otimes_S H' \rightarrow 0.$$

Since H' is a projective T -module, so is $P_i \otimes_S H'$ and hence

$$\text{pdim}((S/WV) \otimes_S H')_T \leq n$$

(for the weak dimension, observe that $P_i \otimes_S H'$ is a flat T -module whenever P_i is a flat S -module).

From the short exact sequence

$$0 \rightarrow WV \rightarrow S \rightarrow S/WV \rightarrow 0$$

we obtain the exact sequence

$$0 \rightarrow WV \otimes_S H' \rightarrow H' \rightarrow (S/WV) \otimes_S H' \rightarrow 0.$$

It follows that $(S/WV) \otimes_S H'$ is isomorphic to H'/WVH' as T -modules. However, $WVH' = \begin{pmatrix} 0 & 0 \\ W & WV \end{pmatrix}$, since $(WV)W = W$. Thus one can see that T/J is isomorphic to H'/WVH' and this proves the result.

Remark 1.5. Clearly it follows from Lemmas 1.3 and 1.4 that if W is a flat S -module and if $(WV)W = W$, then for any right T -module M such that $MJ = 0$ we can conclude that

$$\text{pdim}(M)_T \leq \text{r.gldim}(S/WV) + \text{pdim}(S/WV)_S,$$

and

$$\text{wdim}(M)_T \leq \text{w.gldim}(S/WV) + \text{wdim}(S/WV)_S.$$

From now on we will denote the right side of the first inequality by α , and the right side of the second by α' .

There is a natural setting for when the conditions of Lemma 1.4 are satisfied. Namely, if $VW = R$, then W is a finitely generated projective left S module, since it has a projective basis (let $\sum v_i w_i = 1$ and then let $f_i: W \rightarrow S$ be defined by $f(w) = wv_i$, then the f_i 's and the w_i 's form a projective basis). Similarly, V is a finitely generated projective right S module. Furthermore $(WV)W = W(VW) = W$. This setting will be used in §2.

Lemma 1.6. *Let V be flat as a left R -module. Then for any right R -module N , $\text{pdim}(N \otimes_R H)_T \leq \text{pdim}(N)_R$ and $\text{wdim}(N \otimes_R H)_T \leq \text{wdim}(N)_R$.*

Proof. As in Lemma 1.4 we only have to prove the projective dimension case. Again we can assume that $\text{pdim}(N)_R = n < \infty$. Thus we have a projective resolution of the R -module N

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow N \rightarrow 0.$$

Since H is flat as a left R -module, we have

$$0 \rightarrow P_n \otimes_R H \rightarrow \cdots \rightarrow P_0 \otimes_R H \rightarrow N \otimes_R H \rightarrow 0.$$

As in the proof of Lemma 1.4, it is clear that $P_i \otimes_R H$ is projective as a T -module and hence $\text{pdim}(N \otimes_R H)_T \leq n$.

Before we get to the main result of the section, we need some facts relating the projective and weak dimensions of modules in an exact sequence. For a module M , both dimensions will be denoted $\dim(M)$. Consider the short exact sequence of modules

$$0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0.$$

Then

- (1) If $\dim(A) > \dim(B)$, then $\dim(C) = \dim(A)$
- (2) If $\dim(A) \leq \dim(B)$, then $\dim(C) \leq \dim(B) + 1$.

For the projective dimension results, see, for example, [K, Part III, Theorem 2]. The weak dimension results can be proved in the same fashion using Tor instead of Ext . In particular one can conclude that

$$\dim(C) \leq \max\{\dim(B) + 1, \dim(A)\}.$$

Lemma 1.7. *Let $0 \rightarrow M \rightarrow N \xrightarrow{f} P \rightarrow Q \rightarrow 0$ be an exact sequence of right modules. Then $\dim(P) \leq \max\{\dim(M) + 1, \dim(Q), \dim(N)\}$.*

Proof. Let U be the cokernel of f . By the above,

$$\dim(U) \leq \max\{\dim(M) + 1, \dim(N)\}.$$

The bound on $\dim(P)$ now follows from a standard argument using Tor or Ext .

We are now able to prove the main result of this section, which gives a bound on the dimensions of T .

Theorem 1.8. *Let V and W be flat as left R - and S -modules, respectively, and let $(WV)W = W$. Then*

$$\max\{\text{r.gldim } R, \text{r.gldim } S\} \leq \text{r.gldim}(T) \leq \max\{\alpha + 1, \text{r.gldim}(R)\}$$

and

$$\max\{\text{w.gldim } R, \text{w.gldim } S\} \leq \text{w.gldim}(T) \leq \max\{\alpha' + 1, \text{w.gldim}(R)\},$$

where α and α' are given in Remark 1.5.

Proof. As usual we only prove the global dimension case. The left-hand inequality is Lemma 1.2. For the other inequality, let M be any right T -module. From the short exact sequence

$$0 \rightarrow J \rightarrow T \rightarrow T/J \rightarrow 0$$

we get the exact sequence of right T -modules

$$(*) \quad 0 \rightarrow \operatorname{Tor}_T^1(M, T/J) \rightarrow M \otimes_T J \rightarrow M \rightarrow M \otimes_T (T/J) \rightarrow 0.$$

First we need bounds on the dimension of $M \otimes_T J$. Consider the surjection $f: K \otimes_R H \rightarrow J$ given by multiplication $k \otimes h \rightarrow kh$.

Claim. $(\operatorname{Ker} f) \cap J = 0$.

Let $\sum \begin{pmatrix} r_i & 0 \\ w_i & 0 \end{pmatrix} \otimes \begin{pmatrix} r'_i & v_i \\ 0 & 0 \end{pmatrix}$ be in $\operatorname{ker} f$. Then $\sum r_i r'_i = 0$, $\sum r_i v_i = 0$, $\sum w_i r'_i = 0$, and $\sum w_i v_i = 0$. Let $\begin{pmatrix} r & v \\ w & xy \end{pmatrix} \in J$, where $x \in W$ and $y \in V$. Then

$$\begin{aligned} & \left(\sum \begin{pmatrix} r_i & 0 \\ w_i & 0 \end{pmatrix} \otimes \begin{pmatrix} r'_i & v_i \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} r & v \\ w & xy \end{pmatrix} \\ &= \sum \begin{pmatrix} r_i & 0 \\ w_i & 0 \end{pmatrix} \otimes \begin{pmatrix} r'_i r + v_i w & r'_i v + v_i xy \\ 0 & 0 \end{pmatrix} \\ &= \sum \begin{pmatrix} r_i & 0 \\ w_i & 0 \end{pmatrix} \otimes \begin{pmatrix} r'_i r & r'_i v \\ 0 & 0 \end{pmatrix} + \sum \begin{pmatrix} r_i & 0 \\ w_i & 0 \end{pmatrix} \otimes \begin{pmatrix} v_i w & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \sum \begin{pmatrix} r_i & 0 \\ w_i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & (v_i x)y \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since the tensor product is over R , each of the three sums is zero. This proves the claim.

Clearly the kernel A of the induced surjection

$$M \otimes_T K \otimes_R H \rightarrow M \otimes_T J$$

is also killed by J . By Remark 1.5 $\operatorname{pdim}(A) \leq \alpha$ and by Lemma 1.6

$$\operatorname{pdim}(M \otimes_T K \otimes_R H) \leq \operatorname{r.gldim}(R).$$

So from the discussion prior to Lemma 1.7,

$$\operatorname{pdim}(M \otimes_T J) \leq \max\{\alpha + 1, \operatorname{r.gldim}(R)\}.$$

Now we return to sequence $(*)$. The modules $\operatorname{Tor}_T^1(M, T/J)$ and $M \otimes_T (T/J)$ are killed by J , and hence, by Remark 1.5, both of these modules have projective dimension less than or equal to α . Therefore, by Lemma 1.7, $\operatorname{pdim}(M_T) \leq \max\{\alpha + 1, \operatorname{r.gldim}(R)\}$. Hence the result is proved.

As noted after Remark 1.5, if $VW = R$ and if ${}_R V$ is flat, then the hypotheses of Theorem 1.8 are satisfied. Also in this case T and S are Morita equivalent ([LS, Lemma 1.2]). Thus we have the following results:

$$\operatorname{r.gldim}(S) \leq \max\{\alpha + 1, \operatorname{r.gldim}(R)\}$$

and

$$\text{w.gldim}(S) \leq \max\{\alpha' + 1, \text{w.gldim}(R)\}.$$

If S is any ring and $e \in S$ is an idempotent, then $(eSe, S, eS, Se, \theta, \psi)$, where θ and ψ are multiplication, is a Morita context. So when eS is flat as a left eSe -module, the above result is an improvement on [KK, Proposition 2.6].

2. APPLICATIONS TO IDEALIZERS AND FIXED RINGS

There are two natural examples of Morita contexts to which we want to apply the results of the previous section. For the first, let I be a right ideal of the ring R and let S be a subring of R which contains I as a two-sided ideal. Then S is called a *subidealizer* of I in R . In the notation of the last section, if we let $V = R$ and $W = I$, then

$$T = \begin{pmatrix} R & R \\ I & S \end{pmatrix} \text{ is a Morita context.}$$

We are able to use the results of the last section to relate $\text{w.gldim}(R)$ and $\text{w.gldim}(S)$. There have been a number of papers relating the global dimension of R and S . First Goodearl [G] had some results under the assumption that S/I is semisimple. Teply [T] and Hansen [H] obtained a generalization of this result which eliminated the semisimple assumption. More recently Robson and Small [RS, Theorems 2 and 5] have further improved on this in a number of directions. In our next result we use Theorem 1.8 to give relations between the weak global dimensions of R and S , which are the same as the ones obtained for the right global dimension in [RS, Theorem 5] (note that we can also get their relations from Theorem 1.8). Before we give the theorem let us recall that a right ideal I of R is said to be *generative* if $RI = R$. In this case, using the notation of the last section, we have $VW = R$ and $WV = I$.

Theorem 2.1. *Let I be a generative right ideal of R . If S is a subidealizer of I in R , then*

$$\begin{aligned} \text{w.gldim}(R) &\leq \text{w.gldim}(S) \\ &\leq \max\{\text{w.gldim}(S/I) + \text{wdim}(R/I)_R + 1, \text{w.gldim}(R)\}. \end{aligned}$$

Proof. Since I is generative, as we have noted before, T and S are Morita equivalent. So by Lemma 1.2, $\text{w.gldim}(R) \leq \text{w.gldim}(S)$. Clearly $V = R$ is flat as a left R -module, and since $VW = R$, the hypotheses of Theorem 1.8 are satisfied. By [G, Proposition 1.2], $\text{wdim}(I)_R \leq \text{wdim}(I)_S$. Furthermore, since $VW = R$, $V = R$ is projective as an S -module, hence any flat R -module is a flat S -module by Lazard's Theorem. This gives us the reverse inequality. So $\text{wdim}(S/I)_S = \text{wdim}(R/I)_R$ and thus the result follows from Theorem 1.8.

Finally, as a corollary we obtain an improvement of a result of Goodearl in [G].

Corollary 2.2. *Under the previous assumptions, if in addition S/I is a von Neumann regular ring, then $\text{w.gldim}(R) \leq \text{w.gldim}(S) \leq \text{w.gldim}(R) + 1$.*

Proof. The result follows immediately from the fact that $\text{w.gldim}(S/I) = 0$.

Our second application arises in the Galois theory of noncommutative rings. Let G be a finite group of automorphisms acting on the ring R . The fixed ring R^G is defined to be the subring $\{x \in R \mid x^g = x \text{ for all } g \in G\}$. If $|G|^{-1} \in R$, then, as noted in [L, 2.2], it is well known that

$$(*) \quad \text{r.gldim}(R^G) \leq \text{r.gldim}(R) + \text{pdim}(R)_{R^G}.$$

Lorenz replaced the assumption that $|G|^{-1} \in R$ by other, more general, assumptions on R and G . He then obtains certain bounds on $\text{r.gldim}(R^G)$ when R is flat as a left R^G -module. We stay in the classical case where $|G|$ is a unit in R ; in addition we use Lorenz's assumption that R is flat as a left R^G -module. Then we improve on $(*)$ by showing that

$$\text{r.gldim}(R^G) \leq \text{r.gldim}(R).$$

Let R and G be as before. The *skew group ring* $R * G$ is the set of all formal sums $\sum_{g \in G} r_g g$, $r_g \in R$. Addition is componentwise and multiplication is defined distributively by the formula

$$rg \cdot sh = rs^{g^{-1}}gh,$$

for $r, s \in R$ and $g, h \in G$. Clearly R is a left and right module over R^G . R can also be viewed as a left or right $R * G$ -module as follows: for any $s = \sum r_g g \in R * G$ and $r \in R$, define $s \cdot r = \sum r_g r^{g^{-1}}$ and $r \cdot s = \sum (rr_g)^g$. Then $T = \begin{pmatrix} R^G & R \\ R & R * G \end{pmatrix}$ is a Morita context. See [C] for details. We note that the map $\psi: R \otimes_{R^G} R \rightarrow R * G$ is defined by $\psi(x \otimes y) = \sum_{g \in G} xy^{g^{-1}}g$. So it is not difficult to see that the image of ψ is the two-sided ideal generated by the element $\sum_{g \in G} g$. We will denote this ideal by L . The map $\theta: R \otimes_{R * G} R \rightarrow R^G$ is given by $\theta(x \otimes y) = \sum_{g \in G} (xy)^g$ and if $|G|^{-1} \in R$, then θ is easily seen to be onto, i.e., in the notation of §1, $VW = R^G$.

Theorem 2.3. *If $|G|^{-1} \in R$ and if R is flat as a left R^G -module, then*

$$\text{r.gldim}(R^G) \leq \text{r.gldim}(R) \leq \max\{\alpha + 1, \text{r.gldim}(R^G)\},$$

where $\alpha = \text{r.gldim}(R * G/L) + \text{pdim}(R * G/L)_{R * G}$.

Proof. Since $VW = R^G$, T and $R * G$ are Morita equivalent. Also, by [McCR, Theorem 7.5.6], R is projective as a left $R * G$ -module and $\text{r.gldim}(R * G) = \text{r.gldim}(R)$. So, by Lemma 1.2 and Theorem 1.8, we are done.

Note that the above result does not include a statement about weak global dimension, since the assumptions easily imply, from other considerations, that $\text{w.gldim}(R^G) \leq \text{w.gldim}(R)$ (namely, if $|G|$ is a unit in R , then one can show, as in the global dimension case, that $\text{w.gldim}(R^G) \leq \text{w.gldim}(R) + \text{wdim}(R)_{R^G}$).

In light of the hypothesis in Theorem 2.3 it is natural to ask, when is R flat over the fixed ring R^G ? There does not seem to be a great deal of work on

this question though there are some situations in which this happens naturally. There are a number of cases where R is projective over R^G (see, for example, [L]). Though, in view of Theorem 2.3 and [L, 2.2], it would be more interesting to know when R is merely flat over R^G .

If R is regular, then it is well known that R^G is regular if the order of G is a unit in R . So, in this case, both parts of the hypothesis of Theorem 2.3 are satisfied. Furthermore, Jøndrup [J, Examples 4 and 5] has given examples where R is regular and $|G|^{-1} \in R$, yet R is not projective over R^G . For another example recall that a commutative domain is called a Prufer domain if every finitely generated ideal is invertible (equivalently, projective). Now assume that R is such a domain with quotient field Q . Let I be a finitely generated right ideal of R^G . Since IR is a finitely generated ideal of R , there exists $b_i \in IR$ and $q_i \in (IR)^* = \{q \in Q | IRq \subseteq R\}$, $i = 1, \dots, n$, such that $\sum_1^n b_i q_i = 1$. Since $b_i \in IR$ and since $(IR)^*$ is an R -submodule of Q , we can assume that $b_i \in I$ (though q_i is only in $(IR)^*$). We then get

$$\sum_{g \in G} \left(\sum_1^n b_i q_i^g \right) = |G|1,$$

which, by rearranging terms, becomes

$$\sum_1^n b_i \left(\sum_{g \in G} q_i^g \right) = |G|1.$$

For a fixed i , the element $\sum_{g \in G} q_i^g$ is in Q^G , which, if $|G|^{-1} \in R$, is well known to be the quotient field of R^G . Moreover, this element is in I^* , so if $|G|^{-1} \in R$, it follows that I is invertible, i.e., R^G is also a Prufer domain. Clearly R is torsion free as a module over R^G and, since torsion-free modules over a Prufer domain are necessarily flat, we have another class of examples where R is flat over R^G .

Finally we give a simple example to show that the first inequality in Theorem 2.3 can be strict. Let K be a field of characteristic not two and let g be the automorphism of $K[x]$ which sends $f(x)$ to $f(-x)$. Clearly the ideal I generated by x^2 is g -invariant. Hence g defines an automorphism of $R = K[x]/I$. Let $G = \langle g \rangle$, then $R^G = K$. Clearly R^G has global dimension 0 and R is flat over R^G , yet R has infinite global dimension.

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