ANTIHOOK DIFFERENCES AND SOME PARTITION IDENTITIES

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ABSTRACT. Anti-hook differences are applied to give new combinatorial interpretations to three identities from Slater's Compendium.

1. INTRODUCTION, DEFINITIONS AND THE MAIN RESULTS

Recently in [2] new partition-theoretic parameters called antihook differences were introduced and applied to give new interpretations to the following identities of Rogers [5]:

$$(1.1) \quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_{2n}} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{2n+1})(1 - q^{20n+4})(1 - q^{20n+16})},$$

$$(1.2) \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n+1}} = \prod_{\substack{n=1 \\ n \not\equiv \pm 3, \pm 4, \pm 7, 10 \pmod{20}}}^{\infty} (1 - q^n)^{-1},$$

$$(1.3) \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n}} = \prod_{\substack{n=1 \\ n \not\equiv \pm 1, \pm 8, \pm 9, 10 \pmod{20}}}^{\infty} (1 - q^n)^{-1},$$

$$(1.4) \quad \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q; q)_{2n+1}} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{2n+1})(1 - q^{20n+8})(1 - q^{20n+12})},$$

where $(a; q)_n = \sum_{i=0}^{\infty} (1 - aq^i)/(1 - aq^{n+i})$.

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In this paper we apply antihook differences to give new combinatorial interpretations to the following identities from [6]:

(1.5)

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q)_{2n}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{8n-1})(1 - q^{8n-7})(1 - q^{16n-10})(1 - q^{16n-6})(1 - q^{8n})$$

[6, (39)–(83)],

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q; q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})} \quad [6, (84)],$$

(1.7)

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{8n-3})(1 - q^{8n-5})(1 - q^{16n-14})(1 - q^{16n-2})(1 - q^{8n})$$

[6, (38)–(86)].

We shall use the technique of [3] and some of the results of [1].

Definition 1 [2]. Let Π be a partition whose Ferrers graph is embedded in the fourth quadrant. Each node (i, j) of the fourth quadrant that is not in the Ferrers graph of Π is said to possess an anti-hook difference $\rho_i - k_j$ relative to Π , where ρ_i is the number of nodes on the i th row of the fourth quadrant to the left of node (i, j) that are not in the Ferrers graph of Π , and k_j is the number of nodes in the j th column of the fourth quadrant that lie above node (i, j) and are not in the Ferrers graph of Π .

Definition 2. The nodes (i, j) of Π for which $i - j = a$ are said to lie on diagonal a .

Definition 3 [3]. A partition with “ $n + l$ copies of n ”, $l \geq 0$ is a partition in which a part of size n , $n \geq 0$, can come in $n + l$ different colors denoted by subscripts: $n_1, n_2, \dots, n_{n+l}$.

In the part n_i , n can be zero if and only if $i \geq 1$. But in no partition are zeros permitted to repeat. Thus, for example, the partitions of 2 with “ $n + 1$ copies of n ” are: $2_1, 2_1 + 0_1, 1_1 + 1_1, 1_1 + 1_1 + 0_1, 2_2, 2_2 + 0_1, 1_2 + 1_1, 1_2 + 1_1 + 0_1, 2_3, 2_3 + 0_1, 1_2 + 1_2, 1_2 + 1_2 + 0_1$.

Definition 4 [3]. The weighted difference of two parts m_i, n_j , $m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.

Definition 5 [4]. The rank of a partition is defined as the largest part minus the number of parts.

Definition 6. A right angle in the graphical representation of a partition is called a hook and will be denoted by $[a, b]$ if there are a nodes in the row and b nodes in the column. Thus, for example, $[5, 4]$ represents the hook

and $[1, 1]$ represents the single node $\bullet$.

We shall prove that identities (1.5)–(1.7) have their combinatorial counterparts in the following theorems, respectively.

Theorem 1. Let $A_1(\nu)$ denote the number of partitions of ν that have all anti-hook differences on diagonal 0 equal to 0 or 1 and no hook has all anti-hook differences on diagonal 0 equal to 0, that is, no hook has rank equal to 0. Let $B_1(\nu)$ denote the number of partitions of ν into parts $\not\equiv 0, \pm 1, \pm 6, \pm 7, 8 \pmod{16}$. Then $A_1(\nu) = B_1(\nu)$.

Example. $A_1(9) = 5$, since the relevant partitions are $9, 81, 71^2, 61^3, 531$; also, $B_1(9) = 5$, since in this case the relevant partitions are $54, 52^2, 432, 3^3, 32^3$.

Theorem 2. Let $A_2(\nu)$ denote the number of partitions of ν that have all anti-hook differences on diagonal 0 equal to 0 or 1 and, if $[a, b]$ and $[c, d]$ are any two hooks such that $a > c$ and $b > d$, then $b > c + 1$. Let $B_2(\nu)$ denote the number of partitions of ν into distinct parts. Then $A_2(\nu) = B_2(\nu)$.

Theorem 3. Let $A_3(\nu)$ denote the number of partitions of ν that have all anti-hook differences on diagonal -2 equal to 1 or 2 and no hook has all anti-hook differences on diagonal -2 equal to 2, that is, no hook has rank equal to 2. Let $B_3(\nu)$ denote the number of partitions of ν into parts $\not\equiv 0, \pm 2, \pm 3, \pm 5, 8 \pmod{16}$. Then $A_3(\nu) = B_3(\nu)$.

Remark. It should be noted here that the left sides of (1.5)–(1.7) are the generating functions of $A_k(\nu)$ for $1 \leq k \leq 3$, respectively.

To prove Theorems 1–3 we shall use the following results from [1], in which $B_k(\nu)$, $1 \leq k \leq 3$, are the same as defined above in Theorems 1–3.

Theorem 4 [1, Theorem 1.6]. Let $C_1(\nu)$ denote the number of partitions of ν with “ n copies of n ” such that

- (4.a) each pair of parts has nonnegative weighted difference, and
- (4.b) even parts appear with even subscripts and odd parts with odd subscripts greater than 1.

Then $C_1(\nu) = B_1(\nu)$.

Theorem 5 [1, Theorem 1.4]. Let $C_2(\nu)$ denote the number of partitions of ν with “ n copies of n ” such that

- (5.a) each pair of parts has positive weighted difference, and
- (5.b) even parts appear with even subscripts and odd parts with odd subscripts.

Then $C_2(\nu) = B_2(\nu)$.

Theorem 6 [1, Theorem 1.7]. Let $C_3(\nu)$ denote the number of partitions of ν with “ $n + 2$ copies of n ” such that

- (6.a) for some i , i_{i+2} is a part,
- (6.b) the parts are nonnegative,

- (6.c) *each pair of parts has nonnegative weighted difference, and*
 (6.d) *even parts appear with even subscripts and odd parts with odd subscripts greater than 1.*

Then $C_3(\nu) = B_3(\nu)$.

Our object here is to prove the following result:

Theorem 7. For $1 \leq k \leq 3$,

$$A_k(\nu) = C_k(\nu).$$

2. PROOF OF THEOREM 7

Each of these three cases is proved in a similar way. We provide the details for $k = 1$ and sketch the changes required to treat the remainder.

Let Π be a partition enumerated by $A_1(\nu)$. Let

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$, and $a_1 + a_2 + \cdots + a_r + b_1 + b_2 + \cdots + b_r + r = \nu$, be the corresponding Frobenius's notation [3]. Then the anti-hook difference conditions of Theorem 1 are equivalent to

$$(2.1.a) \quad a_i > b_i$$

and

$$(2.1.b) \quad b_i \geq a_{i+1} + 1.$$

We now establish a 1-1 correspondence between the ordinary partitions enumerated by $A_1(\nu)$ and the partitions with n copies of n enumerated by $C_1(\nu)$. We do this by mapping each column $\begin{smallmatrix} a \\ b \end{smallmatrix}$ of the Frobenius Symbol to a single part m_i of a partition with n copies of n . The mapping ϕ is

$$(2.2) \quad \phi \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{cases} (a+b+1)_{b-a} & \text{if } a < b, \\ (a+b+1)_{a-b+1} & \text{if } a > b. \end{cases}$$

It should be noted here that since Π is enumerated by $A_1(\nu)$ so in the graph of Π there is no hook which has all the anti-hook differences on the diagonal 0 equal to 0. It means that there is no possibility of a being equal to b .

The inverse mapping ϕ^{-1} is given by

$$(2.3) \quad \phi^{-1}: m_i \rightarrow \begin{cases} \begin{pmatrix} (m-i-1)/2 \\ (m+i-1)/2 \end{pmatrix} & \text{if } m \not\equiv i \pmod{2} \\ \begin{pmatrix} (m+i-2)/2 \\ (m-i)/2 \end{pmatrix} & \text{if } m \equiv i \pmod{2}, \quad i \neq 1. \end{cases}$$

Now for any two adjacent columns $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}$ in the Frobenius Symbol with $\phi \begin{pmatrix} a \\ b \end{pmatrix} = m_i$ and $\phi \begin{pmatrix} c \\ d \end{pmatrix} = n_j$ (defined by (2.2)), we have

$$(2.4) \quad ((m_i - n_j)) = \begin{cases} 2b - 2c - 2 & \text{if } a > b, c > d \\ 2a - 2c - 1 & \text{if } a < b, c > d \\ 2b - 2d - 1 & \text{if } a > b, c < d \\ 2a - 2d & \text{if } a < b, c < d. \end{cases}$$

Clearly (2.1.a) and (2.2) imply (4.b) and then (2.1.b) and only the first line of (2.4) imply (4.a).

To see the reverse implication we note that by (4.b) $m \equiv i, n \equiv j \pmod{2}$ and so under ϕ^{-1}

$$(2.5) \quad a - c = \frac{1}{2}((m_i - n_j)) + i,$$

$$(2.6) \quad b - d = \frac{1}{2}((m_i - n_j)) + j,$$

$$(2.7) \quad a - b = i - 1,$$

$$(2.8) \quad b - c - 1 = \frac{1}{2}((m_i - n_j)).$$

Now by (4.a), (2.5) and (2.6) guarantee that $a_i > a_{i+1}$ and $b_i > b_{i+1}$. (2.7) together with the fact that $i > 1$ implies (2.1.a) and (2.8), by (4.a) implies (2.1.b). This completes the proof of $A_1(\nu) = C_1(\nu)$.

For $k = 2$, the anti-hook difference conditions are equivalent to

$$(2.9.a) \quad a_i \geq b_i$$

and

$$(2.9.b) \quad b_i > a_{i+1} + 1.$$

The map ϕ is

$$(2.10) \quad \phi \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{cases} (a + b + 1)_{b-a} & \text{if } a < b, \\ (a + b + 1)_{a-b+1} & \text{if } a \geq b, \end{cases}$$

and ϕ^{-1} is given by

$$(2.11) \quad \phi^{-1}: m_i \rightarrow \begin{cases} \binom{(m-i-1)/2}{(m+i-1)/2} & \text{if } m \not\equiv i \pmod{2}, \\ \binom{(m+i-2)/2}{(m-i)/2} & \text{if } m \equiv i \pmod{2}. \end{cases}$$

For $k = 3$, the anti-hook difference conditions are equivalent to

$$(2.12) \quad b_{i+1} + 3 \leq a_i < b_i + 2, \quad a_i \neq 1.$$

The map ϕ is

$$(2.13) \quad \phi \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{cases} (a + b + 1)_{b-a+3} & \text{if } a < b + 2, \\ (a + b + 1)_{a-b-2} & \text{if } a > b + 2, \quad a \neq 1. \end{cases}$$

and ϕ^{-1} is given by

$$(2.14) \quad \phi^{-1}: m_i \rightarrow \begin{cases} \binom{(m+i+1)/2}{(m-i-3)/2} & \text{if } m \not\equiv i + 2 \pmod{2}, \\ \binom{(m-i+2)/2}{(m+i-4)/2} & \text{if } m \equiv i + 2 \pmod{2}. \end{cases} \quad m \neq i, i \neq 1.$$

To illustrate the constructed bijections we give an example for $k = 1, \nu = 9$

shown in the following table:

Partitions enumerated by $A_1(9)$	Frobenius symbol for partitions enumerated by $A_1(9)$	Image under ϕ (that is, partitions enumerated by $C_1(9)$)
9	$\begin{pmatrix} 8 \\ 0 \end{pmatrix}$	9_9
8 + 1	$\begin{pmatrix} 7 \\ 1 \end{pmatrix}$	9_7
7 + 1 + 1	$\begin{pmatrix} 6 \\ 2 \end{pmatrix}$	9_5
6 + 1 + 1 + 1	$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$	9_3
5 + 3 + 1	$\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$	$7_3 + 2_2$

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