ON A LIPSCHITZ INVARIANT OF NORMED SPACES

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ABSTRACT. C. Bessaga introduced an invariant $\eta(X)$ for σ-compact normed linear spaces. He showed that $\eta(X) = \eta(Y)$ whenever X and Y are Lipschitz homeomorphic. In this note we construct two σ-compact normed spaces with $\eta(X) = \eta(Y)$ which are not Lipschitz homeomorphic. Moreover, there are no compact convex sets K and L generating X and Y, respectively, which are Lipschitz homeomorphic. This answers two problems posed by Bessaga.

Let K be a compact metric space. Denote by $e_n(K)$ the infimum of those $\varepsilon > 0$ so that K contains an ε-net of size 2^n, and let $e(K)$ be the set of all positive sequences (a_n) so that $\lim a_n/e_n(K) = 0$. If X is a σ-comapct metric space, put $\eta(X) = \cap(\cup_n e(A_n))$, where the intersection is taken over all countable compact covers (A_n) of X.

These definitions were introduced by C. Bessaga in [Bes], and used by him to construct a normed linear space X which is not Lipschitz homeomorphic to its closed subspaces of codimension 1. He showed that if X is a closed subset of Y, or is a Lipschitz image of Y, then $\eta(X) \subseteq \eta(Y)$. In particular it follows that $\eta(X) = \eta(Y)$ when X and Y are Lipschitz homeomorphic.

At the end of his article, Bessaga asks if one can construct two non-Lipschitz-homeomorphic spaces for which $\eta(X) = \eta(Y)$ (Problem A), and whether $\eta(X) = \eta(Y)$ implies that there are Lipschitz-homeomorphic compact and convex sets K and L generating X and Y, respectively (i.e., $X = \cup nK$ and $Y = \cup nL$) (Problem B).

In this short article we answer these problems by constructing σ-comapct normed linear spaces with $\eta(X) = \eta(Y)$, so that any compact convex set that generates X cannot be Lipschitz homeomorphic to a subset of Y. Both the construction and the proofs are simple, and we use known techniques for reducing problems of Lipschitz classification to the linear case by differentiation.

We shall use standard notation and terminology (see for example [LT] and the survey article [Ben].

Let $\{p_i : -\infty < i < \infty\}$ and $\{q_i : -\infty < i < \infty\}$ be two doubly infinite

sequences so that $3 < p_i < q_i < p_{i+1} < 4$ for all i, and let

$$Z_p = \left(\sum_i \oplus \ell_{p_i}\right)_2 = \left\{ x_{ij} : \|x_{ij}\| = \left(\sum_i \left(\sum_j |x_{ij}|^{p_i}\right)^{2/p_i}\right)^{1/2} < \infty \right\}.$$

Define similarly $Z_q = (\sum_i \oplus \ell_{q_i})_2$, and put $K = \{x_{ij} : |x_{ij}| \le 2^{-|i|-j}\}$.

K is a compact and convex set when regarded as a subset of either Z_p or Z_q, and we let X_p be its linear span, $X_p = \cup nK$, with the Z_p norm, and X_q is the same linear space with the Z_q norm.

X_p and X_q are σ-compact, and $\eta(X_p) = \eta(X_q)$, because each of these spaces is a Lipschitz image of the other. Indeed, as $p_i < q_i$ for all i, the formal identity is a norm -1 linear operator from X_p into X_q which is obviously onto. Conversely, the map $T(x_{ij}) = (x_{i-1,j})$ is a norm -1 operator from X_q into X_p (because $q_i < p_{i+1}$ for all i). It is also an onto map because $K/2 \subseteq T(K) \subseteq 2K$.

Proposition. *Let L be a compact convex subset of X_p which generates X_p. Then L cannot be Lipschitz homeomorphic to a subset of Z_q.*

To prove the Proposition, we first need a few preliminaries. The first Lemma was used in [Bes, proof of Proposition 2]. We give the proof for the sake of completeness.

Lemma 1. *Let L be a compact convex subset of the normed space X such that $X = \cup nL$, and let K be a compact convex subset of X. Then there is an n_0 such that $K \subseteq n_0 L$.*

Proof. By the Baire Category Theorem, there are n_1, $x_0 \in K$, and $\varepsilon > 0$, so that $K \cap B(x_0, \varepsilon) \subseteq n_1 L$. By the compactness and convexity of K, there is a $\delta > 0$ such that $(1 - \delta)x_0 + \delta k \in K \cap B(x_0, \varepsilon) \subseteq n_1 L$ for all $k \in K$; i.e., $K \subseteq \delta^{-1}(n_1 L + (\delta - 1)x_0) \subseteq n_0 L$, provided n_0 is large enough, by the convexity of L.

Let K be a subset of a Banach space X, and let f be a function from X into another Banach space Y. Fix a point k in K, and let X_k be the subspace of X of all points x such that $k + \lambda x \in K$ for small enough $|\lambda|$. The differential of f at the point k, is a linear operator $Df_k : X_k \to Y$ such that $Df_k(x) = \lim_{\lambda \to 0}(f(k + \lambda x) - f(k))/\lambda$ for all $x \in X_k$.

Note that if f satisfies a Lipschitz condition with constant C, then Df_k (when it exists) is a bounded linear operator of norm at most C. Similarly, if f is a Lipschitz embedding, then Df_k is an into linear isomorphism. Finally, if X_k is dense in X and f is a Lipschitz embedding, then Df_k extends by continuity to a linear isomorphism of all of X into Y.

Our next Lemma is a summary of the results of [M, §4] in the context of Banach spaces, using a somewhat different terminology.

Lemma 2. *Let f be a Lipschitz function from a compact convex subset K of a normed space X into a Banach space Y with the Radon–Nikodym Property.*

Assume K is affinely homeomorphic to the Hilbert cube Q, and let μ be the image on K of the usual measure on Q. Then f has a differential μ-a.e. on K.

The last lemma follows by a routine gliding-hump argument form the fact that l_r is not isomorphic to a subspace of l_s when $r \neq s$ (see [LT]).

Lemma 3. *Z_p is not linearly isomorphic to a subspace of Z_q.*

Proof of proposition. By Lemma 1, L contains a multiple of K. It thus suffices to prove that K, considered as a subset of Z_p, cannot be Lipschitz-embedded into Z_q. But K is affinely homeomorphic to the Hilbert cube and, for almost all points k in K, X_k is dense in Z_p. Also Z_q, being reflexive, has the Radon–Nikodym Property. It thus follows from Lemma 2 and the remarks preceding it that if K were Lipschitz homeomorphic to a subset of Z_q, then Z_p would be linearly isomorphic to a subspace of Z_q—but this contradicts Lemma 3!

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