A SPREAD RELATION FOR ENTIRE FUNCTIONS WITH NEGATIVE ZEROS

FARUK F. ABI-KHUZAM

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ABSTRACT. Let g be a canonical product having only real negative zeros and nonintegral order λ , and let ϕ be the set function defined by $2\pi\phi(E) = \int_E \pi\lambda \csc \pi\lambda \cos \lambda\theta \, d\theta$. It is shown that if $E(r)$ is the set of values of $\theta \in (-\pi, \pi]$ where $|g(re^{i\theta})| \geq 1$, r_n is a sequence of Polya peaks of g and δ is the deficiency of the value zero of g then $\liminf \phi(E(r_n)) \geq 2(1-\delta)^{-1}$. This inequality leads to a sharp spread relation for g .

1. INTRODUCTION

Let

$$(1) \quad g(z) = \prod_{n=1}^{\infty} (1 - z/a_n) \exp \left(\sum_{j=1}^q \frac{1}{j} (z/a_n)^j \right)$$

denote a Weierstrass canonical product of genus $q \geq 1$, having only negative zeros a_n . Denote by

$$(2) \quad T(r, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |g(re^{i\theta})| \, d\theta$$

the Nevanlinna characteristic of g , and denote by

$$(3) \quad \mu = \liminf_{r \rightarrow \infty} \frac{\log T(r, g)}{\log r}$$

the lower order of g .

Let

$$(4) \quad E(r) = \{\theta \in [-\pi, \pi]: \log |g(re^{i\theta})| \geq 0\},$$

and denote by $|E(r)|$ the measure of the Lebesgue measurable set $E(r)$. Then it is a consequence of the spread relation for meromorphic functions conjectured by Edrei and proved by Baernstein [3] that

$$(5) \quad \liminf_{m \rightarrow \infty} |E(r_m)| \geq \frac{\pi}{\mu},$$

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where r_m is a sequence of Polya peaks of order μ of $T(r, g)$. The exact meaning of these peaks will not be needed here; for their definition see [4]. In private communication, Professor Daniel Shea asked whether (5) can be improved for the functions g defined by (1) and having only negative zeros. It is the purpose of this note to obtain a sharp form of (5) for such functions.

Recall that the deficiency of the value 0 of g is defined by

$$(6) \quad \delta(0) = \delta(0, g) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, 1/g)}{T(r, g)},$$

where

$$(7) \quad N(r, 1/g) = \int_0^r t^{-1} n(t) dt$$

and $n(t)$ is the number of zeros of g in the disk $|z| \leq t$. Our results may be stated as follows:

Theorem 1. *Let g be a Weierstrass canonical product of genus $q \geq 1$, having only negative zeros and assume that the lower order μ of g is nonintegral. Let ψ_μ be the function defined on $(-\pi, \pi)$ by*

$$(8) \quad \psi_\mu(\theta) = \pi\mu \csc \pi\mu \cos \mu\theta,$$

and introduce the set function ϕ defined on Lebesgue measurable subsets of $[-\pi, \pi]$ by

$$(9) \quad \phi(E) = \frac{1}{\pi} \int_E \psi_\mu(\theta) d\theta.$$

If r_m is a sequence of Polya peaks of order μ of $T(r, g)$ and $\delta(0)$ is the deficiency of the value 0 of g , then

$$(10) \quad \liminf_{m \rightarrow \infty} \phi(E(r_m)) \geq \frac{2}{1 - \delta(0)}$$

and

$$(11) \quad \liminf_{m \rightarrow \infty} |E(r_m)| \geq \frac{2(q+1)}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi\mu|}{(q+1)(1-\delta(0))} \right\}.$$

Furthermore, both inequalities are sharp.

If the lower order of g satisfies $q < \mu \leq q + \frac{1}{2}$, then it may happen that one component of the set $E(r_m)$ extends all the way up to π . In this case we can obtain a sharper form of (11):

Theorem 2. *Let g be as in Theorem 1 and assume that*

$$(12) \quad q < \mu \leq q + \frac{1}{2}.$$

If $\pi \in E(r_m)$ for infinitely many values of m , then

$$(13) \quad \limsup_{m \rightarrow \infty} |E(r_m)| \geq 2\pi(1 - q/\mu) + \frac{2q}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi\mu| \delta(0)}{q(1 - \delta(0))} \right\}.$$

Furthermore, (13) is sharp.

For the case $0 < \mu < 1$, the analogues of (10) and (11) continue to hold true [4] and require no geometric restriction on the zeros of g . As for (13), the spread relation of Edrei gives the stronger result $\liminf |E(r_m)| = 2\pi$.

An entire function having finite lower order μ and having all its zeros on one ray necessarily [2] has finite order λ and satisfies $\lambda \leq [\mu] + 1$. Thus for the functions g defined by (1) we always have $q \leq \mu \leq \lambda \leq q+1$. The assumptions in Theorems 1 and 2 imply $q < \mu < q+1$.

Our proofs depend on the following two results of Hellerstein and Williamson [5]:

Theorem A. Let g be as in the statement of Theorem 1 and let

$$(14) \quad C(r) = \{\theta \in [0, \pi]: \log |g(re^{i\theta})| \geq 0\}.$$

Then for each $r > 0$, there exist points $\alpha_1 = \alpha_1(r), \dots, \alpha_{q+1} = \alpha_{q+1}(r)$ satisfying

$$(15) \quad \frac{2j-1}{2(q+1)}\pi < \alpha_j < \frac{2j-1}{2q}\pi \quad (j = 1, 2, \dots, q)$$

and

$$(16) \quad \frac{2q+1}{2(q+1)}\pi < \alpha_{q+1} \leq \pi,$$

such that

$$(17) \quad C(r) = \bigcup_{j=1}^{(q+1)/2} [\alpha_{2j-1}, \alpha_{2j}] \quad \text{if } q \text{ is odd}$$

and

$$(18) \quad C(r) = \bigcup_{j=0}^{q/2} [\alpha_{2j}, \alpha_{2j+1}], \alpha_0 = 0 \quad \text{if } q \text{ is even}.$$

Theorem B. Let g be as in Theorem 1 and let r_m be a sequence of Polya peaks of order μ of $T(r, g)$. If

$$(19) \quad \sigma > \limsup_{r \rightarrow \infty} \frac{N(r, 1/g)}{T(r, g)},$$

then

$$(20) \quad 1 \leq \sigma(1 + o(1)) |\csc \pi \mu| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) + o(1)$$

as $m \rightarrow \infty$, where $\alpha_{q+2} = 0$.

The inequality in (20) above is a restatement of the inequality (4.17) in [5, p. 249] where we have divided by $T(r_m, g)$. The dependence on the Polya peaks occurs in $\alpha_j = \alpha_j(r_m)$.

2. PROOF OF THEOREM 1

The sets $E(r)$ and $C(r)$ defined in (4) and (14) respectively satisfy $E(r) = C(r) \cup -C(r)$. This is because the function $|g(re^{i\theta})|$ is an even function of θ , a fact which follows from (1) and the reality of the zeros a_n . Recalling the definitions of $\psi_\mu(\theta)$ and ϕ we have

$$(2.1) \quad \frac{1}{2}\phi(E(r_m)) = \frac{1}{\pi} \int_{C(r_m)} \psi_\mu(\theta) d\theta.$$

If q is odd, then by (17) the integral in (2.1) equals

$$\csc \pi \mu \sum_{j=1}^{(q+1)/2} (\sin \mu \alpha_{2j} - \sin \mu \alpha_{2j-1}) = |\csc \pi \mu| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}),$$

where we have used $\alpha_0 = \alpha_{q+2} = 0$ and $q < \mu < q+1$ to write $\csc \pi \mu = -|\csc \pi \mu|$. If q is even, then in view of (18) the integral in (2.1) equals

$$\csc \pi \mu \sum_{j=0}^{q/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) = |\csc \pi \mu| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}).$$

Using (20) and the above, we obtain

$$(2.2) \quad 2 \leq \sigma(1 + o(1))\phi(E(r_m)) + o(1) \quad \text{as } m \rightarrow \infty.$$

Now (10) follows immediately from (2.2), (19), and (6).

We note here that (2.2) and (2.1) imply

$$(2.3) \quad \begin{aligned} 2 &\leq \sigma(1 + o(1))\phi(E(r_m)) + o(1) \\ &\leq \sigma(1 + o(1)) \times \frac{2}{\pi} \int_0^\pi \psi_\mu^+(\theta) d\theta + o(1) \quad \text{as } m \rightarrow \infty. \end{aligned}$$

The first and last inequalities imply that

$$(2.4) \quad \liminf_{r \rightarrow \infty} \frac{T(r, g)}{N(r, 1/g)} \leq \frac{1}{\pi} \int_0^\pi \psi_\mu^+(\theta) d\theta,$$

which is the well-known result of Hellerstein and Williamson [5]. Note that (2.3) may be used to obtain information about functions extremal for the inequality (2.4). Indeed, if equality holds in (2.4), then (2.3) implies that $\phi(E(r_m)) \rightarrow \frac{1}{\pi} \int_0^\pi \psi_\mu^+(\theta) d\theta$ as $m \rightarrow \infty$, and we may conclude from this that

$$\alpha_j(r_m) \rightarrow \frac{(2j-1)}{2\mu} \pi \quad \text{as } m \rightarrow \infty,$$

for $j = 1, \dots, q+1$ if $q + \frac{1}{2} < \mu \leq q+1$, and for $j = 1, \dots, q$ and $\alpha_{q+1} \rightarrow \pi$ if $q < \mu \leq q + \frac{1}{2}$.

Proof of (11). Suppose first that q is odd. Using the elementary identity $\sin x - \sin y = 2 \cos(\frac{x+y}{2}) \sin(\frac{x-y}{2})$, we obtain

$$\begin{aligned}
 & |\csc \pi \mu| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) \\
 &= |\csc \pi \mu| \sum_{j=1}^{q+1} (-1)^{j-1} \sin \mu \alpha_j \\
 (2.5) \quad &= 2 |\csc \pi \mu| \sum_{j=1}^{(q+1)/2} \sin \frac{1}{2} \mu (\alpha_{2j-1} - \alpha_{2j}) \cos \frac{1}{2} \mu (\alpha_{2j-1} + \alpha_{2j}) \\
 &\leq 2 |\csc \pi \mu| \sum_{j=1}^{(q+1)/2} \sin \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}).
 \end{aligned}$$

To justify the last inequality in (2.5), we must show that $\frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \leq \pi$ for $j = 1, 2, \dots, (q+1)/2$. By (15) and (16) we have $\alpha_{q+1} - \alpha_q \leq \pi - \frac{2q-1}{2(q+1)} \pi = \frac{3\pi}{2(q+1)} < \frac{2\pi}{\mu}$ since $q < \mu < q+1$. Also, if $q > 1$, then for $j = 1, \dots, (q-1)/2$ we have by (15)

$$\begin{aligned}
 \alpha_{2j} - \alpha_{2j-1} &\leq \frac{4j-1}{2q} \pi - \frac{4j-3}{2(q+1)} \pi = \frac{4j+2q-1}{2q(q+1)} \pi \\
 &\leq \frac{4q-3}{2q(q+1)} \pi < 2\pi/(q+1) < 2\pi/\mu.
 \end{aligned}$$

Thus in all cases $\frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \leq \pi$ and the inequality in (2.5) is justified.

To continue, note that the function $\sin x$ is concave on $[0, \pi]$ since $(\sin x)'' = -\sin x \leq 0$ for all $x \in [0, \pi]$. Thus $\sin(n^{-1} \sum_{j=1}^n x_j) \geq n^{-1} \sum_{j=1}^n \sin x_j$ for any n numbers $x_j \in [0, \pi]$. Taking $x_j = \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1})$ where $j = 1, \dots, (q+1)/2$, we obtain

$$\begin{aligned}
 (2.6) \quad \sum_{j=1}^{(q+1)/2} \sin \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) &\leq \frac{q+1}{2} \sin \left\{ \frac{2}{q+1} \sum_{j=1}^{(q+1)/2} \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \right\} \\
 &= \frac{(q+1)}{2} \sin \left\{ \frac{\mu}{q+1} |C(r_m)| \right\},
 \end{aligned}$$

where $|C(r_m)|$ is the measure of the set $C(r_m)$ defined in (14), (17). Now (20), (2.5), and (2.6) imply

$$(2.7) \quad 1 \leq \sigma(1 + o(1)) \left\{ \frac{(q+1)}{|\sin \pi \mu|} \sin \left(\frac{\mu}{q+1} |C(r_m)| \right) \right\} + o(1)$$

as $m \rightarrow \infty$.

We may assume that for all large m , $|C(r_m)| < (q+1)\pi/2\mu$, since otherwise the inequality in (11) is manifestly true. Thus (2.7) now implies that

$$(2.8) \quad |C(r_m)| \geq \frac{q+1}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi \mu|}{(q+1)} \cdot \frac{(1 + o(1))}{\sigma} \right\} \quad (m \rightarrow \infty).$$

Recalling (19) and (6), we see that (2.8) leads to

$$(2.9) \quad \liminf_{m \rightarrow \infty} |E(r_m)| \geq \frac{2(q+1)}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi \mu|}{(q+1)(1-\delta(0))} \right\},$$

and the proof of (11) is complete when q is odd.

If q is an even integer, we have to proceed slightly differently: Since $[(q+1)/2] = q/2$, the first sum in (2.5) becomes

(2.10)

$$\begin{aligned} & |\csc \pi \mu| \sum_{j=0}^{q/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) \\ &= |\csc \pi \mu| \left\{ \sin \mu \alpha_1 + \sum_{j=1}^{q/2} 2 \cos \frac{1}{2} \mu (\alpha_{2j+1} + \alpha_{2j}) \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) \right\} \\ &\leq |\csc \pi \mu| \left\{ \sin \mu \alpha_1 + 2 \sum_{j=1}^{q/2} \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) \right\}, \end{aligned}$$

where the inequality is justified as in the case of odd q . We now let $x_0 = \mu \alpha_1$, $x_j = \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j})$ for $j = 1, \dots, q/2$ and $x_{-j} = x_j$. This gives us $q+1$ numbers in $[0, \pi]$ to which we may apply the concavity of $\sin x$. We thus obtain

$$\sin \left\{ (q+1)^{-1} \left(x_0 + \sum_{j=1}^{q/2} (x_j + x_{-j}) \right) \right\} \geq (q+1)^{-1} \left\{ \sin x_0 + 2 \sum_{j=1}^{q/2} \sin x_j \right\}.$$

It follows that

$$\begin{aligned} & \sin \mu \alpha_1 + 2 \sum_{j=1}^{q/2} \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) \\ (2.11) \quad & \leq (q+1) \sin \left\{ \frac{\mu \alpha_1 + \sum_{j=1}^{q/2} \mu (\alpha_{2j+1} - \alpha_{2j})}{q+1} \right\} \\ &= (q+1) \sin \left\{ \frac{\mu}{q+1} |C(r_m)| \right\}. \end{aligned}$$

Now (2.10) and (2.11) lead us back to (2.8) and (2.9). Hence (11) is true when q is odd, and the proof of Theorem 1 is complete except for the statement about the sharpness of the inequalities.

3. PROOF OF THEOREM 2

Assume that the lower order μ of g satisfies

$$(3.1) \quad q < \mu \leq q + \frac{1}{2},$$

and that for infinitely many values of m , $\pi \in E(r_m)$. By taking a subsequence of r_m which we also denote by r_m , we may assume that $\pi \in E(r_m)$ for all m . This means that we have

$$(3.2) \quad \alpha_{q+1}(r_m) = \pi \quad \text{for all } m.$$

Now suppose that q is even. Since $\alpha_0 = 0$, we may write

$$(3.3) \quad \begin{aligned} & |\csc \pi \mu| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) \\ & \leq |\csc \pi \mu| \left\{ 2 \sum_{j=1}^{q/2} \sin \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) + \sin \pi \mu \right\} \\ & \leq 1 + 2 |\csc \pi \mu| (q/2) \sin \left\{ \frac{2}{q} \sum_{j=1}^{q/2} \frac{1}{2} \mu (\alpha_{2j} - \alpha_{2j-1}) \right\} \\ & = 1 + q |\csc \pi \mu| \sin \left\{ \frac{\mu}{q} \sum_{j=1}^{q/2} (\alpha_{2j} - \alpha_{2j-1}) \right\}. \end{aligned}$$

Recalling (18) and (3.2) we have, with $r = r_m$,

$$(3.4) \quad \sum_{j=1}^{q/2} (\alpha_{2j} - \alpha_{2j-1}) = \alpha_{q+1} - |C(r)| = \pi - |C(r)|.$$

Now (3.3) and (3.4) used in (20) give

$$(3.5) \quad 1 \leq \sigma(1 + o(1)) \cdot \left(1 + q |\csc \pi \mu| \sin \left\{ \frac{\pi \mu}{q} - \frac{\mu}{q} |C(r_m)| \right\} \right) + o(1)$$

as $m \rightarrow \infty$.

Since $0 \leq y \leq \sin x$ for some x in $[0, \pi]$ implies $\sin^{-1} y \leq \pi - x$, we conclude from (3.5), with $x = \frac{\mu}{q}(\pi - |C(r_m)|)$, that

$$\sin^{-1} \left\{ \frac{|\sin \pi \mu|}{q} \left(\frac{1 + o(1)}{\sigma(1 + o(1))} - 1 \right) \right\} \leq \pi - (\pi - |C(r_m)|) \frac{\mu}{q};$$

and this readily yields (13) when q is even.

If q is odd, then (17) and (3.2) imply that

$$(3.6) \quad \sum_{j=1}^{(q+1)/2} (\alpha_{2j} - \alpha_{2j-1}) = \pi - \sum_{j=0}^{(q-1)/2} (\alpha_{2j+1} - \alpha_{2j}).$$

Since $\alpha_0 = \alpha_{q+2} = 0$, the sum appearing in (20) may be written

$$\begin{aligned}
 (3.7) \quad & |\csc \pi \mu| \sum_{j=0}^{[(q+1)/2]} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) \\
 &= |\csc \pi \mu| \left\{ \sin \mu \alpha_1 + \sum_{j=1}^{(q-1)/2} (\sin \mu \alpha_{2j+1} - \sin \mu \alpha_{2j}) + |\sin \pi \mu| \right\} \\
 &\leq |\csc \pi \mu| \left\{ \sin \mu \alpha_1 + 2 \sum_{j=1}^{(q-1)/2} \sin \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j}) + |\sin \pi \mu| \right\} \\
 &\leq |\csc \pi \mu| \left\{ q \sin \left\{ \frac{\mu \alpha_1 + 2 \sum_{j=1}^{(q-1)/2} \frac{1}{2} \mu (\alpha_{2j+1} - \alpha_{2j})}{q} \right\} + |\sin \pi \mu| \right\} \\
 &= |\csc \pi \mu| \left\{ q \sin \left\{ \frac{\mu}{q} (\pi - |C(r_m)|) \right\} + |\sin \pi \mu| \right\} \\
 &= 1 + q |\csc \pi \mu| \sin \left\{ \frac{\pi \mu}{q} - \frac{\mu}{q} |C(r_m)| \right\}.
 \end{aligned}$$

Thus the inequality (3.5) also holds true in the case of odd q , and hence (13) follows in this case also. Note that if $q = 1$, the sum in the second line in (3.7) is empty.

4. THE BEST POSSIBLE CHARACTER OF THEOREMS 1 AND 2

Let g be a Weierstrass canonical product of genus $q \geq 1$ and order μ , $q < \mu < q + 1$, having only real negative zeros and satisfying

$$(4.1) \quad N(r) = N(r, 1/g) \sim r^\mu L(r) \quad \text{as } r \rightarrow \infty,$$

where L is a slowly varying function. It is known [1] that

$$(4.2) \quad \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\log |g(re^{i\theta})|}{N(r)} - \psi_\mu(\theta) \right|^p d\theta = 0 \quad (1 \leq p < \infty).$$

It follows from (4.2) that

$$(4.3) \quad \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\log^+ |g(re^{i\theta})|}{N(r)} - \psi_\mu^+(\theta) \right| d\theta = 0,$$

and also that

$$(4.4) \quad \lim_{r \rightarrow \infty} \left| \int_{E(r)} \left(\frac{\log |g(re^{i\theta})|}{N(r)} - \psi_\mu(\theta) \right) d\theta \right| = 0,$$

where $E(r)$ is any measurable subset of $[-\pi, \pi]$.

Now (4.3) implies that $\lim_{r \rightarrow \infty} T(r, g)/N(r) = \frac{1}{\pi} \int_0^\pi \psi_\mu^+(\theta) d\theta$. If we take $E(r)$ to be the set defined in (4), then (4.4) implies

$$\begin{aligned} \lim_{r \rightarrow \infty} \phi(E(r)) &= \lim_{r \rightarrow \infty} \frac{1}{\pi} \int_{E(r)} \frac{\log |g(re^{i\theta})|}{N(r)} d\theta \\ (4.5) \qquad \qquad \qquad &= \lim_{r \rightarrow \infty} \frac{2T(r, g)}{N(r)} = \frac{2}{1 - \delta(0)}, \end{aligned}$$

and this shows that equality can hold in (10).

From (4.5) and (4.1), it follows that $T(r, g)$ is regularly varying and so every sequence is a sequence of Polya peaks of order μ of $T(r, g)$. Thus, by the remark after the proof of (2.4), we conclude that for such functions g ,

$$(4.6) \qquad \qquad \qquad \alpha_j(r) \rightarrow \frac{(2j-1)}{2\mu} \pi \quad \text{as } r \rightarrow \infty$$

for $j = 1, \dots, (q+1)$ if $q + \frac{1}{2} \leq \mu < q+1$; and for $j = 1, \dots, q$ and $\alpha_{q+1}(r) \rightarrow \pi$ if $q < \mu \leq q + \frac{1}{2}$. It follows that for such functions g , the measure of the set $E(r)$ defined in (4) satisfies

$$(4.7) \qquad \qquad \qquad |E(r)| \rightarrow |E_\mu|,$$

where $E_\mu = \{\theta \in [-\pi, \pi]: \psi_\mu(\theta) \geq 0\}$.

We can also compute $\delta(0)$:

$$\frac{1}{1 - \delta(0)} = \frac{1}{\pi} \int_0^\pi \psi_\mu^+(\theta) d\theta = \begin{cases} \frac{q+1}{|\sin \pi \mu|} & \text{if } q + \frac{1}{2} \leq \mu < q+1 \\ \frac{q + |\sin \pi \mu|}{|\sin \pi \mu|} & \text{if } q < \mu \leq q + \frac{1}{2}. \end{cases}$$

It follows that

$$(4.8) \qquad |E_\mu| = \frac{(q+1)\pi}{\mu} = \frac{2(q+1)}{\mu} \sin^{-1} \left\{ \frac{|\sin \pi \mu|}{(q+1)(1 - \delta(0))} \right\} \quad \text{if } q + \frac{1}{2} < \mu < q+1.$$

And if $q < \mu < q + \frac{1}{2}$, then

$$\begin{aligned} |E_\mu| &= \frac{(q-1)\pi}{\mu} + 2 \left(\pi - \frac{(2q-1)\pi}{2\mu} \right) \\ &= 2\pi \left(1 - \frac{q}{\mu} \right) + \frac{(q-1)\pi}{\mu} + \frac{\pi}{\mu} \\ &= 2\pi \left(1 - \frac{q}{\mu} \right) + \frac{q}{2\mu} \sin^{-1} \left(\frac{|\sin \pi \mu| \delta(0)}{q(1 - \delta(0))} \right). \end{aligned}$$

Now (4.7) and (4.8) imply that equality can hold in (13), which must therefore be a best possible inequality.

REFERENCES

1. F. F. Abi-Khuzam, *An abelian theorem for a class of subharmonic functions*, Proc. Amer. Math. Soc. **67** (1977), 253–259.
2. —, *The order of entire functions with radially distributed zeros*, Proc. Amer. Math. Soc. **82** (1981), 71–75.

3. A. Baernstein II, *Proof of Edrei's spread conjecture*, Proc. London Math. Soc. **26** (1973), 418–434.
4. A. Edrei, *The deficiencies of meromorphic functions of finite lower order*, Duke Math. J. **31** (1964), 1–22.
5. S. Hellerstein and J. Williamson, *Entire functions with negative zeros and a problem of R. Nevanlinna*, J. Analyse Math. **22** (1969), 233–267.

DEPARTMENT OF MATHEMATICAL SCIENCES, KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS, DHAHRAN 31261, SAUDI ARABIA

Current address: Department of Mathematics, American University of Beirut, Beirut, Lebanon