BRANGESIAN SPACES IN THE POLYDISK

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ABSTRACT. In this paper we extend to the polydisk D^2 a theorem of L. de Branges which characterizes the class of all Hilbert spaces that are contractively contained in the classical Hardy space H^2 of the disk and which are invariant under the shift S acting as an isometry. Our theorem characterizes Hilbert spaces which are vector subspaces of $H^2(D^2)$ and which are invariant under the operators of multiplication by the coordinate functions whose actions are isometric and which doubly commute. We do not use contractivity.

1. INTRODUCTION

In [2] L. de Branges has proved a theorem which not only extends Beurling's [1] famous shift-invariant subspace theorem but also its vector-valued generalizations due to Peter Lax [5] and Paul Halmos [3]. The theorem of de Branges (scalar version) replaces the shift-invariant subspaces of Beurling's theorem by a class of Hilbert spaces of analytic functions which are contractively contained in the Hardy space H^2 of the open unit disk D and on which multiplication by the coordinate function z is an isometry.

In this paper we give a two-variable version of de Branges's theorem on the space $H^2(D^2)$ of analytic functions on the bidisk D^2 . In fact, we establish something more general by dropping the contractivity condition. This gives us a larger class of Hilbert spaces from which de Branges-type spaces can be obtained easily as a particular case. We shall also show that the necessary and sufficient condition for Beurling-type theorems to be valid in D^2 [6] follows as a simple consequence. However, as was kindly pointed out to us by the referee, this consequence is a straightforward deduction from the much more elementary and far earlier work of Radlow [7].

2. TERMINOLOGY AND PRELIMINARY RESULTS

Let D be the open unit disk in the complex plane and T be its boundary, i.e. the unit circle. Let D^2 and T^2 be the Cartesian products of D and T respectively. Let m denote the normalized two-dimensional Lebesgue measure

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on T^2 , so that if $T^2 = \{(e^{i\theta_1}, e^{i\theta_2}) : 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi\}$, then $dm = d\theta_1 d\theta_2 / 4\pi^2$. For $p > 0$, let $L^p(T^2, d\theta_1 d\theta_2)$ be the Lebesgue spaces on T^2 and let $H^p(D^2)$ denote the class of both variable analytic functions on D^2 such that

$$\sup_{0 \leq r \leq 1} \frac{1}{4\pi^2} \int_{T^2} |f(re^{i\theta_1}, re^{i\theta_2})|^p d\theta_1 d\theta_2 < \infty.$$

By means of the Poisson kernel $P[(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})] = P_{r_1}(\theta_1 - \theta_2) \cdot P_{r_2}(\theta_1 - \theta_2)$ (where $P_r(\theta) = 1 - r^2 / (1 - 2r \cos \theta + r^2)$), one can identify $H^p(D^2)$ with a closed subspace of $L^p(T^2, d\theta_1 d\theta_2)$. In particular, $H^2(D^2)$ is identified with the class of all $L^2(T^2, d\theta_1 d\theta_2)$ functions $f(e^{i\theta_1}, e^{i\theta_2})$ which have Fourier series of the type $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} e^{im\theta_1} e^{in\theta_2}$ such that $\|f\|_2 = (\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{mn}|^2)^{1/2} < \infty$. We denote this class by $H^2(T^2)$, which is a Hilbert space under the above norm. $H^\infty(T^2)$ stands for the class of all essentially bounded functions in $H^2(T^2)$, which is a Banach algebra under the essential supremum norm. For further information we refer to [8].

A bounded linear operator S on a Hilbert space H is said to be an isometry if $\|Sx\|_H = \|x\|_H$ for all x in H . Let S_1 and S_2 be the operators on $H^2(T^2)$ given by

$$S_k f(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_k} f(e^{i\theta_1}, e^{i\theta_2}) \quad (k = 1, 2).$$

Obviously S_1 and S_2 are isometries on $H^2(T^2)$. It is also easy to see that S_1 commutes with S_2^* (the adjoint of S_2) on $H^2(T^2)$. Two commuting operators A, B on a Hilbert space H are said to be doubly commuting if A commutes with B^* , or equivalently if B commutes with A^* . An isometry S on H is said to be a shift if $\bigcap_{n=0}^{\infty} S^n(H) = \{0\}$.

Let G be a semigroup and let $S(t)$, $t \in G$, be an abelian semigroup of isometries on a Hilbert space H . A closed subspace L of H is said to be a wandering subspace for $S(t)$ if $S(t_1)L \perp S(t_2)L$ for every t_1, t_2 in G , $t_1 \neq t_2$. We shall need the following results from [12].

Lemma 2.1. Let V_1, V_2 be two doubly commuting isometries on a Hilbert space H which are shifts. Then

- (i) The semigroup $\{V_1^m V_2^n\}_{m,n \geq 0}$ is of type 's' (see [12, p. 255]) and
- (ii) $L_1 \cap L_2$ is a wandering subspace for the semigroup $\{V_1^m V_2^n\}_{m,n \geq 0}$ and $H = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \oplus V_1^m V_2^n (L_1 \cap L_2)$, where $L_k = H\theta V_k(H)$ ($k = 1, 2$).

Proof. See [12, Theorem 1].

Lemma 2.2. Let V_1, V_2 be commuting isometries on the Hilbert space H . If $L_1 \cap L_2 = 0$, where $L_k = H\theta V_k H$, then the semigroup $\{V_1^m V_2^n\}_{m,n \geq 0}$ is not of the type 's'.

Proof. See [12, Corollary 1].

A Hilbert space H is said to be contractively contained in a Hilbert space K if:

- (i) H is a vector subspace of K and
- (ii) $\|x\|_K \leq \|x\|_H$ for each x in H .

3. THE THEOREM OF DE BRANGES ON THE UNIT CIRCLE

Let g be any function in the unit ball of $H^\infty(T)$ and let

$$M(g) = \{g(e^{i\theta})f(e^{i\theta}) : f(e^{i\theta}) \in H^2(T)\}.$$

(For details on these classical $H^p(T)$ spaces, see [4] or [10]). $M(g)$ becomes a Hilbert space under the norm $\|gf\|_M = \|f\|_2$. It is easy to verify that $M(g)$ is contractively contained in $H^2(T)$ and that the shift operator, i.e. the operator of multiplication by the coordinate function $e^{i\theta}$, acts as an isometry on $M(g)$, which is invariant under the shift. The theorem of de Branges [9] says the converse:

Theorem 3.1. *Let M be a Hilbert space contractively contained in H^2 such that M is invariant under the shift, which acts as an isometry on M . Then there is a unique g in the unit ball of $H^\infty(T)$ (unique up to a factor of modulus one) such that $M = M(g)$.*

In [11] we have observed that dropping the contractivity condition enables us to prove the more general result below. Let g be any element of H^∞ and let $N(g) = \{gf : f \in H^2(T)\}$. $N(g)$ is a Hilbert space under the norm $\|gf\|_N = \|f\|_2$. It is easy to see that $N(g)$ is contractively contained in H^2 if and only if g is in the unit ball of H^∞ . It is also obvious that multiplication by the coordinate function $e^{i\theta}$ leaves $N(g)$ invariant and that it is an isometry on $N(g)$. The following is its converse.

Theorem 3.2. *Let N be a Hilbert space which is a vector subspace of $H^2(T)$ and such that multiplication by the coordinate function $e^{i\theta}$ leaves N invariant and is an isometry on N . Then there exists a g in H^∞ (unique up to a factor of modulus one) such that $N = N(g)$.*

Proof. See [11].

4. THE MAIN RESULT

Let $g(e^{i\theta_1}, e^{i\theta_2})$ be in $H^\infty(T^2)$ and let

$$N(g(e^{i\theta_1}, e^{i\theta_2})) = \{g(e^{i\theta_1}, e^{i\theta_2})f(e^{i\theta_1}, e^{i\theta_2}) : f \in H^2(T^2)\}.$$

$N(g)$ becomes a Hilbert space under the norm $\|gf\|_N = \|f\|_2$ where $\|f\|_2 = (\sum_{m=0}^\infty \sum_{n=0}^\infty |a_{mn}|^2)^{1/2}$ for $f(e^{i\theta_1}, e^{i\theta_2}) = \sum_{m=0}^\infty \sum_{n=0}^\infty a_{mn} e^{im\theta_1} e^{in\theta_2}$. It is also easy to verify that $S_k(N(g)) \subset N(g)$ ($k = 1, 2$), where $S_k f = e^{i\theta_k} f(e^{i\theta_1}, e^{i\theta_2})$ and that S_1, S_2 are doubly commuting isometries on $N(g)$. The converse is our main result.

Theorem. Let N be a Hilbert space which is a vector subspace of $H^2(T^2)$ such that $S_k(N) \subset N$ ($k = 1, 2$) and S_1, S_2 are doubly commuting isometries on N . Then there exists a g in $H^\infty(T^2)$ unique up to a factor of modulus one such that $N = N(g)$.

We first derive two consequences.

Corollary 1. Let N be as in the above theorem and let it be contractively contained in $H^2(T^2)$. Then g lies in the unit ball of $H^\infty(T^2)$.

Proof. Since $N = N(g)$ is contractively contained in $H^2(T^2)$, we have

$$\begin{aligned}\|gf\|_2 &\leq \|gf\|_N \\ &= \|f\|_2 \quad (\because \|gf\|_N = \|f\|_2).\end{aligned}$$

Hence the norm of the operator of multiplication by g is less than or equal to one. But this is easily seen to be the $H^\infty(T^2)$ norm of g . Hence g lies in the unit ball of $H^\infty(T^2)$.

Corollary 2 [6, Theorem 2]. Let N be a closed subspace of $H^2(T^2)$ such that N is invariant under S_1, S_2 , which doubly commute and acts as isometries on N . Then there is an inner function q (unique up to a factor of modulus one) such that $N = qH^2(T^2)$.

Proof. It is obvious by the theorem that $N = N(q) = qH^2(T^2)$ for a unique q in the unit ball of $H^\infty(T^2)$. Further, because N has the same norm as that of $H^2(T^2)$, it follows that multiplication by $q(e^{i\theta_1}, e^{i\theta_2})$ is an isometry on $H^2(T^2)$. Thus $|q(e^{i\theta_1}, e^{i\theta_2})| = 1$ a.e., i.e. q is an inner function.

Remark. As was pointed out to us by the referee, Theorem 2 of [6] is an easy consequence of the much more elementary and far earlier work of Radlow [7].

Proof of the theorem. It is easy to see that $\bigcap_{n=0}^\infty S_k^n(N) = \{0\}$ ($k = 1, 2$). Hence S_1 and S_2 are shifts on N . Setting $L_k = N \ominus S_k(N)$ ($k = 1, 2$), we get by Lemma 2.1 that $L_1 \cap L_2$ is a wandering subspace for the semigroup $\{S_1^m S_2^n\}_{m, n \geq 0}$, and by Lemma 2.2 that $L_1 \cap L_2 \neq \{0\}$. Hence

$$(1) \quad N = \sum_{m=0}^\infty \sum_{n=0}^\infty \oplus S_1^m S_2^n (L_1 \cap L_2).$$

Let us choose a nonzero g in $L_1 \cap L_2$ such that $\|g\|_N = 1$. Then $\{e^{im\theta_1} e^{in\theta_2} g\}_{m, n \geq 0}$ is an orthonormal sequence in N . Let

$$f(e^{i\theta_1}, e^{i\theta_2}) = \sum_{m=0}^\infty \sum_{n=0}^\infty a_{mn} e^{im\theta_1} e^{in\theta_2}$$

be any element of $H^2(T^2)$ and put

$$f_{mn} = \sum_{k=0}^m \sum_{l=0}^n a_{kl} e^{ik\theta_1} e^{il\theta_2}.$$

Then it is clear that $\{f_{mn}\}$ converges to f in $H^2(T^2)$ as m and n tend to infinity by rectangles to be particular. Thus $\{f_{mn}\}_{m,n \geq 0}$ is a Cauchy sequence in $H^2(T^2)$. Now

$$\begin{aligned} \|f_{mn}\|_2 &= \left(\sum_{k=0}^m \sum_{l=0}^n |a_{kl}|^2 \right)^{1/2} \\ &= \left(\sum_{k=0}^m \sum_{l=0}^n \|a_{kl}g \cdot e^{ik\theta_1} e^{il\theta_2}\|_N^2 \right)^{1/2} \\ &= \left\| \sum_{k=0}^m \sum_{l=0}^n (a_{kl}g \cdot e^{ik\theta_1} e^{il\theta_2}) \right\|_N \\ &= \left\| g \sum_{k=0}^m \sum_{l=0}^n a_{kl} e^{ik\theta_1} e^{il\theta_2} \right\|_N \\ &= \|gf_{mn}\|_N. \end{aligned}$$

Hence $\{gf_{mn}\}_{m,n \geq 0}$ is a Cauchy sequence in N . As N is a Hilbert space, it follows that there is an h in N such that $\{gf_{mn}\}$ converges to h in N . Thus

$$h = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g \cdot a_{mn} e^{im\theta_1} e^{in\theta_2};$$

so that letting $g = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} e^{im\theta_1} e^{in\theta_2}$, we have for any fixed positive m and n ,

$$(2) \quad \begin{aligned} h &= a_{00}g + a_{01}e^{i\theta_2}g + a_{10}e^{i\theta_1}g + \cdots + a_{mn}e^{im\theta_1}e^{in\theta_2}g \\ &\quad + e^{i(m+1)\theta_1}h_1 + e^{i(n+1)\theta_2}h_2, \end{aligned}$$

where

$$h_1 = a_{0,n+1}g + a_{0,n+2}e^{i\theta_2}g + \cdots$$

and

$$h_2 = a_{m+1,0}g + a_{m+2,0}e^{i\theta_1}g + \cdots.$$

It is easily seen that h_1 and h_2 are in N and hence in $H^2(T^2)$. Thus from the equation (2) above, we get that the (m, n) th Fourier coefficient of h is the same as (m, n) th Fourier coefficient of the formal series, which is the formal product of the series of g and f . This means that $g(e^{i\theta_1}, e^{i\theta_2}) \cdot f(e^{i\theta_1}, e^{i\theta_2})$ is in N and hence in $H^2(T^2)$. Further, $gf = h$. Since $\|h\|_N = \lim_{m,n} \|gf_{mn}\|_N = \lim_{m,n} \|f_{mn}\|_2 = \|f\|_2$, we conclude that for each f in $H^2(T^2)$, gf is in N and $\|gf\|_N = \|f\|_2$. We next show that g is in $H^\infty(T^2)$ to establish $N(g) \subset N$. By the closed graph theorem, multiplication by g is easily seen to be a bounded linear operator on $H^2(T^2)$, because $gf \in N \subset H^2(T^2)$ for

each f in $H^2(T^2)$. We can assume, without any loss of generality, that the norm of this operator is one. Let $t > 1$ be arbitrarily chosen, and let $I = \{(e^{i\theta_1}, e^{i\theta_2}): |g| > t\}$. Then for each integer $n > 0$,

$$\begin{aligned} 1 &\geq \left(\frac{1}{4\pi^2} \int_{T^2} |g|^{2n} \cdot 1 d\theta_1 d\theta_2 \right)^{1/2} \\ &\geq \left(\frac{1}{4\pi^2} \int_I |g|^{2n} \cdot 1 d\theta_1 d\theta_2 \right)^{1/2} \\ &\geq t^n \frac{1}{2\pi} \int_I d\theta_1 d\theta_2. \end{aligned}$$

As $t > 1$, this can only happen when the Lebesgue measure of I is zero. Thus g is in $H^\infty(T^2)$.

We have thus far shown that $gH^2 \subset N$, $\|gf\|_N = \|f\|_2$, and g is in $H^\infty(T^2)$, i.e. $N(g) \subset N$. To complete the proof of the theorem, we must show that $N(g) = N$ and g is unique up to a factor of modulus one.

We shall first establish that $N(g) = N$. To do this it is sufficient in view of the equality (1) to show that $L_1 \cap L_2$ is one-dimensional. So suppose there is a g_1 in $L_1 \cap L_2$ such that $g_1 \perp g$. Then, by the same methods that were used for g and in view of the equality (1) as well as the facts that S_1 and S_2 are isometries, we conclude that $N(g_1) \subset N$ and $N(g_1) \perp N(g)$. But then gg_1 is in $N(g_1)$ as well as $N(g)$. This forces $gg_1 = 0$ and hence $g_1 = 0$.

For the uniqueness, suppose there is a g_1 in N such that $N = N(g) = N(g_1)$. Then there are elements

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} e^{im\theta_1} e^{in\theta_2} \quad \text{and} \quad h = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} e^{im\theta_1} e^{in\theta_2}$$

such that $g = g_1 h$ and $g_1 = gf$. From this, we deduce that $\|g\|_N = 1 = \|g_1 h\|_N = \|h\|_2$. Similarly $\|f\|_2 = 1$. Further, we have $hf = 1$. Hence from $\|h\|_2 = 1 = \|f\|_2$ we conclude that $|a_{00}| \leq 1$, $|b_{00}| \leq 1$, and from $hf = 1$ we conclude that $|a_{00}b_{00}| = 1$. This means that $1 = |a_{00}| = |b_{00}|$. So that from $\|h\|_2 = 1$, we get $h = a_{00}$ and $g = a_{00}g_1$. This completes the proof of the theorem as $|a_{00}| = 1$.

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