A SHORT SOLUTION OF MARKOV'S PROBLEM ON CONNECTED GROUP TOPOLOGIES

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ABSTRACT. In 1945 A. A. Markov posed a problem concerning the existence of connected group topologies that was recently solved by V. G. Pestov. In this note, a short and independent solution is given.

All topological groups are assumed to be Hausdorff. For a Russian article, the cited translation is used.

In [4] A. A. Markov discussed the existence of connected group topologies. He called a subset M of an abstract group G *unconditionally closed* in G if M is closed with respect to any group topology on G . (Note that this property is called "absolutely closed" in [5].) The following fact can be shown easily [4, Theorem 2.9, p. 270]: If a group G has an unconditionally closed proper subgroup of index less than $2^{\aleph_0}$, then G does not admit a connected group topology.

Now Markov asked whether the inverse theorem is true [4, Problem 5, p. 271]. Recently V. G. Pestov [5] has answered this old question in the negative. For the construction of his counterexample, Pestov develops a complicated technique of independent interest (cf. [5, §§1 and 2]). In this note, quite different examples are given independently by a short proof (see theorem). In March 1989 the author reported on this result at Sofia (Bulgaria). Two weeks later, D. Dikranjan informed him of Pestov's paper.

Let M be an infinite set. Then $S(M)$ denotes the group of all bijections from M onto M . $S(M)$ is called the *symmetric group* of M .

Theorem. *Let M be a set with $|M| \geq 2^{\aleph_0}$. Then every unconditionally closed proper subgroup of $S(M)$ is of index not less than $2^{\aleph_0}$, but $S(M)$ admits only totally disconnected group topologies.*

Proof. By [2] $S(M)$ has no proper subgroups of index less than $|M|$. Let τ_M be the topology of pointwise convergence on $S(M)$ with respect to the discrete topology on M . It is easy to see that τ_M is a group topology which has, as a subbasis for the neighbourhoods of the identity, the collection of all subgroups

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of the form $E_x = \{f \in S(M) : f(x) = x\}$, where $x \in M$ (cf. [1, §7.1]). Then [3, Theorem 2] implies that on $S(M)$ every group topology τ is finer than τ_M . Since τ_M is totally disconnected, τ has the same property.

It is natural to pose the following:

Question. Is there an abelian group G such that G does not admit a connected group topology, but every unconditionally closed proper subgroup of G is of index not less than $2^{\aleph_0}$?

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