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LINKS NOT CONCORDANT TO BOUNDARY LINKS

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ABSTRACT. Casson-Gordon invariants are used to prove that certain links in S^3 are not concordant to boundary links. These examples were first described by Cochran and Orr.

Tim Cochran and Kent Orr recently announced the construction of 2 component links in S^3 with the property that, although all the Milnor $\overline{\mu}$-invariants vanish, the links are not concordant to boundary links [CO]. They also provide examples of higher dimensional links of two components which are not concordant to boundary links. Their 3-dimensional examples include the links $L_m (m > 0)$ illustrated in Figure 1 (see p. 1131). Both bands on the Seifert surface for K_1 are untwisted; that is, the Seifert form vanishes on the homology classes represented by x and y. They let K be the trefoil. The knot K_1 (with $m = 1$) was used in my paper with Gilmer [GL1] as an example of a slice knot that is algebraically doubly null concordant, but not doubly null concordant. The results of [CO] and [GL1] are related. We will show here that the result that L_m is not concordant to a boundary link follows readily from the results of Gilmer [G] on which the work in [GL1] was based. Gilmer's work was based on earlier work on knot concordance by Levine [L] and Casson and Gordon [CG1, CG2].

Danny Ruberman [R1] generalized the work of [GL1] to higher dimensions. (See also [R2] and [Sm] for corrections to [R1].) His observation that the Casson-Gordon invariants apply to higher dimensional slice problems is relevant here. However, a more careful analysis of the Casson-Gordon invariants is needed to prove that the high dimensional examples of [CO] are not concordant to boundary links. That analysis will be presented in a separate paper being written with Pat Gilmer [GL2].

Tim Cochran has informed me that he is also investigating the relationship between the work announced in [CO] and the Casson-Gordon invariant.

The rest of this note is devoted to the proof that L_m is not concordant to a boundary link.

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Pat Gilmer proved that if a slice knot K in S^3 bounds a genus 1 Seifert surface F, then there is a simple closed curve J on F representing a primitive element in $H_1(F)$ and for which Tristram p-signatures, $\sigma_{(s/p)}(J)$, vanish for all $0 < s < p$, where p is any prime dividing the order of the first homology of the 2-fold cyclic cover of S^3 branched over K. Gilmer produced J as follows. Suppose that K bounds a disk D in B^4. The surface $F \cup D$ bounds a 3-manifold R in B^4. Let i denote the inclusion of F into R, and let H denote $i_*^{-1}(\mathrm{Tor}(H_1(R))$. Gilmer showed that H is a 1-dimensional summand of $H_1(F)$ and that if J represents a generator of H then it has the desired signature properties.

To proceed, consider the link $L = K_1 \cup K_2$. K_1 is clearly slice. (Perform ambient surgery on the unknotted curve y on the evident Seifert surface F_1.) We will show that if L is concordant to a boundary link then K_1 bounds a disk D in B^4 with the property that $F_1 \cup D$ bounds a 3-manifold R for which the generator of the summand H described above is represented by x.

The construction of D and R is done in two steps. First, suppose that L is concordant in $S^3 \times [0, 1]$ to the boundary link $L' = K_1' \cup K_2'$ bounding disjoint surfaces F_1' and F_2'. Denote the two components of the concordance C_1 and C_2. The following transversality argument shows that the closed surface $F_1 \cup F_1' \cup C_1$ bounds a 3-manifold R_1 in $S^3 \times [0, 1]$ with R_1 disjoint from C_2. (This was stated without proof in [S].)

Let ν denote tubular neighborhood. There is a map p of $(S^3 \times \{0\} - \nu(K_1 \cup K_2)) \cup (S^3 \times \{1\} - \nu(K_1' \cup K_2')) \cup \partial(\nu(C_2))$ to S^1 such that p is transverse to 1, $p^{-1}(1) = F_1 \cup F_1'$ and $p^{-1}(-1)$ contains $\partial(\nu(C_2))$. Constructing R_1 via transversality depends on extending p to $S^3 \times [0, 1] - \nu(C_1 \cup C_2)$.

Homotopy classes of maps of a space X to S^1 correspond to elements of $H^1(X)$. The restriction map of

$$H^1(S^3 \times [0, 1] - \nu(C_1 \cup C_2))$$

to

$$H^1((S^3 \times \{0\} - \nu(K_1 \cup K_2)) \cup (S^3 \times \{1\} - \nu(K_1' \cup K_2')) \cup \partial(\nu(C_2)))$$

is a map of $\mathbf{Z}^2$ to $\mathbf{Z}^3$ with image consisting of those elements that agree on the meridians of K_1 and K_1'. Our map p represents a class in the image, and hence p is the restriction of a map as desired.

Let D_1 be a slice disk for K_1' in B^4, and let R_1' be a 3-manifold bounded by $F_1' \cup D_1$ in B^4. Form the union of $S^3 \times [0, 1]$ with B^4 identifying $S^3 \times \{1\}$ with ∂B^4. The union of C_1 with D_1 forms a slice disk D for K_1. The union of R_1 and R_1' forms a 3-manifold R bounded by $F_1 \cup D$. Note that R is in the complement of the connected surface $E = C_2 \cup F_2'$ bounded by K_2.

If an element z in $H_1(F_1)$ represents torsion in $H_1(R)$, it is also torsion in $H_1(B^4 - E)$. The element z can be written as $z = ax + by$. In $H_1(B^4 - E)$,

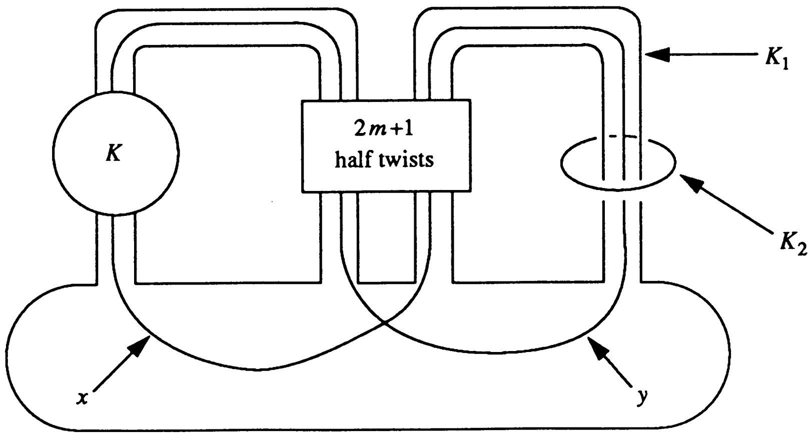

FIGURE 1

x represents zero and y an element of infinite order, in fact the generator. (Linking numbers of x and y with E in B^4 equal their linking numbers with K_2 in S^3.) It follows that b is 0, and hence that H is generated by the class represented by x.

The curve x is of the same knot type as K. The order of the homology of the 2-fold cyclic branched cover of S^3 branched over K_1 is $(2m+1)^2$. Hence, as long as $\sigma_{(s/p)}(K) \neq 0$ for some prime divisor p of $(2m+1)$ and $0 < s < p$, L will not be concordant to a boundary link. The trefoil works for all $m > 0$.

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